

Introduction

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The theoretical foundation for the program of research on which this book is based comes from recognition that individual learning takes place within a community. The members of that community have access to and are influenced by the ideas of others. Individual learners are interconnected with other members of the community; engagement with others opens up possibilities for sharing and comparing representations of ideas and for revising existing schemes and building new ones. In the activity of problem solving, learners bring forth, communicate, and compare ideas. They explore whether the ways that others represent ideas correspond with their own representations, thereby extending their personal repertoires of tools for dealing with new ideas. In this way further learning takes place and understanding deepens (Davis & Maher, 1997; Maher, Martino & Alston, 1993; Maher & Davis, 1990).

The data for this book come from a long-term program of research detailing the collective building of mathematical ideas, which we call the longitudinal study. In this book, we explore student work in one of the mathematics strands of the longitudinal study: counting and combinatorics. It investigates how students' reasoning evolved from elementary and high school years to college.

The reasoning of learners is documented by their actions – that is, what they do, say, build, and write – as they work on strands of tasks. In studying how participants make sense of the complexity of problems, we trace the representations they share, the heuristics they invent and apply, and the modifications they make in building arguments and in offering justifications for solutions.

The authors of the constituent 17 chapters relate how an ordinary group of school children manifest over a twelve-year period an extraordinary array of mathematical ideas that they discursively build and how – with time – their ideas modify and mature as they reason and justify their ideas. The book reports episodes from a long-term study of how mathematical ideas and ways of reasoning are built by students over time. The study has produced over 4500 hours of video, over several sites, involving far more data than can be presented here. However we have selected narratives that feature the voices of several children, as interpreted by a variety of researchers, to weave together a bigger story about how students can educate us about the multifaceted nature of mathematical development. In an important sense, the really big story is still being written as our work in preserving and further analyzing those 4500 hours of video through the Video Mosaic Collaborative¹ continues to reveal new narratives. We invite readers to view the videos at <http://www.video-mosaic.org/>. Along with the narratives offered in the book chapters that follow, these videos enable readers to trace in detail the development of counting/combinatorics ideas and ways of reasoning of learners over more than a decade. Thus, while only some of the children's voices appear in this book, we are indebted to *all* of them for sharing their developing mathematical ideas over time and in divergent contexts, which we continue to study and consider how these children's extraordinary

¹ The Video Mosaic Collaborative is a research and development project sponsored by the National Science Foundation (award DRL-08...) that is leveraging the Rutgers Community Repository to preserve the unique video collection amassed by The Robert B. Davis Institute for Learning at Rutgers University through two decades of research with over four millions dollars of grant funding from the NSF (awards MDR-9053597, REC-9814846, REC-03... and DRL-07...). In addition to preserving the video collection, new tools are being developed for conducting design research and an empirical study that use the videos in the context of teacher education. The editors gratefully acknowledge this considerable support from the National Science Foundation and wish to clarify that all views expressed in this book are those of the authors are not necessarily those of the NSF.

mathematical reasoning may inspire the fields of mathematics education, teacher education, and the learning sciences.

To structure a story that emerges from the chapters, the editors have divided this book into four sections. The two chapters of the first section, respectively, provide historical background of the research study from which the details of the later chapters emerge and describe the design of the study. The first chapter describes the study and the purpose of the research, how the study began, and the conditions under which the research was conducted. It also briefly describes the mathematical ideas and ways of reasoning that emerged from the study. (The details are presented in later chapters.) The second chapter presents the method of the study, its design, including selection of participants, data collection, and analysis, as well as the strand of tasks on which participants were invited to work. The chapter also discusses the importance of the task design for helping learners to develop ways of reasoning.

The second section of the book contains five chapters. These chapters chronicle the work of the study's participants over a seven-year period from grades 2 to 8, tracing the development of their mathematical ideas, heuristics, and forms of reasoning. In particular, the reader will learn how the participating children represented their ideas; developed schemes and strategies; reasoned in specific ways; built inductive arguments; reasoned by cases and by recursion; and connected numbers in Pascal's Triangle to results of previous problems. The authors of Chapter 3 discuss how young children use representations to express their mathematical ideas while building a solution to a particular counting problem (the shirts and jeans problem, described fully in Appendix A, along with all combinatorics problems discussed herein). They show how children structure their representations in response to requests to justify their problem solution and build convincing arguments to early counting problems. The authors of Chapters 4 and 5 discuss students' work on different versions of the towers problems (which involve determining how many towers can be built of various heights when selecting from cubes of various numbers of colors). They show the emergence of different forms of reasoning (cases, contradiction, recursion, and induction) and how, motivated by the need to find the sample space for a basic probability exploration, students revisit the inductive argument for building towers. Chapter 6 discusses how participants collaboratively build representations that help them use reasoning by cases and by recursion to develop justifications for their solutions to classes of pizza problems. (Pizza problems involve determining how many pizzas it is possible to make when selecting from various numbers of toppings and under various other constraints.) Completing this section of the book, Chapter 7 presents the results of an interview with thirteen-year-old Stephanie, who discusses the relationship between the towers problems and the binomial expansion, including how the towers answers can be found in Pascal's Triangle.

The six chapters of the book's third section closely examine the mathematical work of the research participants during their high school years. It shows how the students built important connections using sophisticated mathematical reasoning. In these chapters, the story revolves around the students' proof making, use of representations, acquisition of standard notation, and forging of conceptual connections among isomorphic problems. Specifically, Chapter 8 shows that as they revisit their representations and arguments, students refine representations and clarify arguments. In Chapter 9, students working in groups on towers problems are seen to find and generalized formulas, using methods including controlling for variables, justification by cases, and induction. Chapter 10 shows how a tenth-grade student's binary notation helped his group form connections among the pizza and towers problems, the binomial expansion, and Pascal's Triangle.

Chapter 11 details how representations are a source for making connections in solutions to pizza and tower problems, resulting in the students mapping the structure of the solution of these problems to Pascal's Triangle and how their increasingly sophisticated use of representations led to further development of mathematical reasoning and justification. Chapter 12 discusses how students moved from personal to standard notations in order to express in general form their understanding of solutions to the pizza and towers problems and to extend their understanding in creating an isomorphism from the numerical results in those problems to Pascal's Triangle. The chapter also shows how the students' understanding of extensions of the pizza and tower problems led to their understanding of the addition rule for Pascal's Triangle. The final chapter of section three, Chapter 13, reveals how as high school seniors, days before graduation, the students used their understanding of relationships between the pizza and towers problems and Pascal's Triangle to solve a third isomorphic problem – the Taxicab Problem. (This problem involves finding the number of routes from the starting point – the taxicab stand – to various points on a rectangular

grid.) They recognized the isomorphism, used it to make conjectures about the new problem, saw the need to prove their conjectures, and provided a convincing argument. This chapter concludes the examination of the extraordinary mathematical accomplishments of an ordinary group of students.

The last section of the book, consisting of four chapters, takes stock and looks forward. Chapter 14 examines the epistemological growth of the students, viewed from their own perspectives, challenges some received views, and supports more recent views of how students make sense of mathematics.

Chapter 15 examines a different student population—college undergraduates—and their work with the set of combinatorics problems. The chapter shows that when adult college students are asked to justify ideas and make convincing arguments, an understanding of mathematical reasoning, proof, and generalization can emerge. In Chapter 16, we compare the strategies developed by children and older learners for solving the combinatorics problems and discuss the implications for adult learning.

In closing, Chapter 17 presents the epistemological and methodological contributions of the book. We argue that students must be actively and purposely engaged in their learning so as to take ownership of and be proud of their accomplishments. Mathematics educators need to create opportunities for students to engage in this manner. We have shown that in a program of carefully selected tasks, with minimal intervention by educators who pay careful attention to students' arguments and justifications, students can perform mathematically at high levels. In addition to developing mathematical competency, students who participated in the study gained confidence and a sense of empowerment. They learned to trust their own mathematical ability and they did not rely on outside authority for validation. This confidence and sense of empowerment carried over outside their mathematical work; these students found that the skills they gained through their participation in the longitudinal study helped them succeed in many other areas of study and employment.