APPENDIX A: COMBINATORICS PROBLEMS

Listed here are the major combinatorics problems the students encountered from elementary school through high school, along with brief discussions of solutions.

1. Shirts and Jeans (May 1990, Grade 2; October, 1990, Grade 3) – Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

   He can make six different outfits; each of the two pairs of jeans can be matched with each of the three shirts. The outfits are: blue jeans/white shirt, blue jeans/blue shirt, blue jeans/yellow shirt, white jeans/white shirt, white jeans/blue shirt, and white jeans/yellow shirt.

2. Shirts and Jeans Extended (October, 1990, Grade 3) – Suppose Stephen had another pair of jeans, a black pair. How many different outfits can he now make?

   He can make 12 different outfits: number of shirts times number of jeans.

3. Four-Tall Towers (October 1990, Grade 3; December 1992, Grade 5) – Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

   At each position in the tower, there are two color choices. Therefore, there are $2\times2\times2\times2=16$ possible towers that are four cubes tall. This can be generalized to an n-tall tower with two colors to choose from; there are $2\times2\times2\ldots\times2=2^n$ possible towers that are n cubes tall, when there are two colors to choose from. This can also be generalized to an n-tall tower with m colors to choose from; there are $m\times m\times m\ldots\times m=m^n$ possible towers that are n cubes tall with m colors to choose from. In the following discussions, we will call the first generalization (the n-tall tower with two colors) the towers problem, and we will call the second generalization (the n-tall tower with m colors) the generalized tower problem.

4. Cups, Bowls, and Plates (April 1991, Grade 3) – Pretend that there is a birthday party in your class today. It's your job to set the places with cups, bowls, and plates. The cups and bowls are blue or yellow. The plates are blue, yellow, or orange. Is it possible for 10 children at the party each to have a different combination of cup, bowl, and plate? Show how you figured out the answer to this question.

   Each of the two cup choices can be matched with each of the two bowl choices, and each cup-bowl pair can be matched with any of the three different plate choices. Therefore, there are $2\times2\times3=12$ possibilities. Therefore, yes, it is possible for 10 children at the party each to have a different combination of cup, bowl, and plate.

5. Relay Race (October 1991, Grade 4) – This Saturday there will be a 500-meter relay race at the high school. Each team that participates in the race must have a different uniform (a uniform consists of a solid colored shirt and a solid colored pair of shorts). The colors available for shirts are yellow, orange, blue, or red. The colors for shorts are brown, green, purple, or white. How many different relay teams can participate in the race?

   There are four choices for shirts and four choices for shorts, so there are $4\times4=16$ ways to make uniforms. Sixteen different relay teams can participate.

6. Five-Tall Towers (February 1992, Grade 4; December 1997, Grade 10) – Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes tall.

   There are 2^5=32 towers five cubes tall.

7. Four-Tall Towers with Three Colors (February 1992, Grade 4) – Your group has three colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from three colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

   Since there are three choices for each of four positions, there are 3^4=81 possible towers that are four cubes tall when selecting from three colors.

8. A Five-Topping Pizza Problem (December 1992, Grade 5; December 1997, Grade 10) – Consider the pizza problem, focusing on the number of pizza combinations that can be made when selecting from among five different toppings.

   There are 2^5=32 different pizzas.

9. Guess My Tower (February 1993, Grade 5) – You have been invited to participate in a TV Quiz Show and the opportunity to win a vacation to Disney World. The game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red, and yellow. You are given the following possibilities for a winning tower:

All cubes are exactly the same color.
There is only one red cube.
Exactly two cubes are red.
At least two cubes are yellow.

Which choice would you make and why would this choice be better than any of the others?

In order to decide which is the best choice, we need to find the probability of each choice. The total number of 3-tall towers is 8. The probabilities are:

All cubes are exactly the same color: There are two ways (all red or all yellow). The probability is 2÷8=0.25.

There is only one red cube: There are three ways; the red cube can be on the top, in the middle, or on the bottom. The probability is 3÷8=0.375.
Exactly two cubes are red: This is the same as saying exactly one cube is yellow. The probability is the same as for exactly one red cube: 3÷8=0.375.

At least two cubes are yellow: This is equivalent to saying that either exactly two cubes are yellow or exactly three cubes are yellow. As discussed above, the probability that exactly two cubes are yellow (the same as the probability that exactly two cubes are red) is 0.375. Since there is one way for exactly three cubes to be yellow, that probability is 1÷8=0.125. The probability of either event is therefore 0.375 + 0.125 = 0.5. (We can add because the two events are mutually exclusive.)

"At least two cubes are yellow" is the most likely event.

Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?

The total number of 4-tall towers is 24=16. The probabilities are:

> All cubes are exactly the same colo*r*: There are two ways (all red or all yellow). The probability is 2÷16=0.125.
>
> There is only one red cube: There are four ways; the red cube can be on the top, second from the top, second from the bottom, or on the bottom. The probability is 4÷16=0.25.
>
> Exactly two cubes are red: The number of ways to accomplish this is C(4,2)=6. The probability is therefore 6÷16=0.375.
>
> At least two cubes are yellow: This means that exactly two cubes are yellow, exactly three cubes are yellow, or exactly four cubes are yellow. As discussed above, the probability that exactly two cubes are yellow (the same as the probability that exactly two cubes are red) is 6÷16=0.375. The probability that exactly three cubes are yellow is the same as the probability that one cube is red: 4÷16=0.25. Since there is one way for exactly four cubes to be yellow, that probability is 1÷16=0.0625. The probability of any one of the three events is therefore $0.375 + 0.25 + 0.0625 = 0.6875$.

"At least two cubes are yellow" is the most likely event.

10. The Pizza Problem with Halves (March 1993, Grade 5) – A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pizza or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

    With two topping choices, there are four possibilities for the first half pizza, because each topping can be either on or off that half of the pizza. The four choices are: plain (sausage off, pepperoni off), sausage (sausage on, pepperoni off), pepperoni (sausage off, pepperoni on), and sausage/pepperoni (sausage on, pepperoni on). Consider each of the four possibilities in turn.

    > Case 1: Plain. There are four possibilities for the other half of the pizza, the four listed above (plain, sausage, pepperoni, and sausage/pepperoni).
    >
    > Case 2: Sausage. There are three possibilities for the other half of the pizza: sausage, pepperoni, and sausage/pepperoni. (We omit plain, because we already accounted for the plain-sausage pizza in Case 1.)
    >
    > Case 3: Pepperoni. There are two possibilities remaining for the other half of the pizza: pepperoni and sausage/pepperoni. (Plain and sausage are already accounted for.)
    >
    > Case 4: Sausage/pepperoni. There is only one possibility left for the other half of the pizza; that is sausage/pepperoni.

There are 4+3+2+1=10 possible pizzas with halves.

11. The Four-Topping Pizza Problem (April 1993, Grade 5) – A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.
    There are 2×2×2×2=16 possible pizzas.

12. Another Pizza Problem (April 1993, Grade 5) – The pizza shop was so pleased with your help on the first problem that they have asked us to continue our work. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. The pizza shop now wants to offer a choice of crusts: regular (thin) or Sicilian (thick). How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.
    Each of the 16 4-topping pizzas has two choices of crust, so there are 32 pizzas.

13. A Final Pizza Problem (April 1993, Grade 5) – At customer request, the pizza shop has agreed to fill orders with different choices for each half of a pizza. Remember that they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom, and pepperoni. There is a choice of crusts: regular (thin) and Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other than you have accounted for all possible choices.
    The first half of the pizza can have $2^4 = 16$ possible topping configurations, as described above. Consider each of those configurations in turn. Following the procedure described above for the 2-topping half-pizza problem, we find that there are 16+15+14+…+3+2+1 possible pizzas; this sum is given by16×17÷2. Since each pizza can have a thick or thin crust, we multiply by 2. The number of possible pizzas is16×17÷2×2=272.

14. Counting I and Counting II (March 1994, Grade 6) – How many different two-digit numbers can be made from the digits 1, 2, 3, and 4? Each of four cards is labeled with a different numeral: 1, 2, 3, and 4. How many different two-digit numbers can be made by choosing any two of them?
    Counting I: Assuming that you are not permitted to reuse digits, there are four choices for the first digit and three for the second digit, giving 12 two-digit numbers. (They are 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, and 43.)
    Counting II: There are four choices for the first digit and four choices for the second digit. This makes 16 different two-digit numbers. (They are 11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, and 44.)

15. Towers-Binomial Relationship (March 1996, Grade 8) – In an interview, Stephanie discusses the relationship between the towers problems and the binomial coefficients.
    Binomial coefficients arise in connection with the binomial expansion formula $(a+b)^n$. The following can be shown by induction:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

    The coefficient of $a^{n-r}b^r$ is given by:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This number is the r^{th} entry in the n^{th} row of Pascal's Triangle, and it gives the number of towers with exactly r cubes of one color, when building towers that are n-tall and there are two colors to choose from. Hence, the binomial expansion and the towers problem are isomorphic, with the number of instances of a in the r^{th} term being equal to the number of towers having exactly r cubes.

16. Five-Tall Towers with Exactly Two Red Cubes (January 1998, Grade 10) – You have two colors of Unifix cubes (red and yellow) to choose from. How many five-tall towers can you build that contain exactly two red cubes?

    You are selecting two items (the positions of the two red cubes) from five choices (the number of cubes in the tower); there are 10 ways to do this:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$$

17. Ankur's Challenge (January 1998, Grade 10) – Find all possible towers that are four cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

    Suppose the colors are red, blue, and green. We are counting the towers in three cases: 1) those with two red cubes, one blue cube and one green cube, 2) those with one red cube, two blue cubes, and one green cube, and 3) those with one red cube, one blue cube, and one green cube. The following equation gives the number of ways of selecting m groups of objects of size r_1 through r_m:

$$\binom{n}{r_1, r_2, \ldots, r_m} = \frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_m!}, \text{ where } \sum r_i = n$$

    So the number of 4-tall towers containing exactly two red cubes, one blue cube, and two green cubes is:

$$\binom{4}{2,1,1} = \frac{4!}{2! \cdot 1! \cdot 1!} = 12$$

    Similarly for the other two cases:

$$\binom{4}{1,2,1} = \binom{4}{1,1,2} = 12$$

Hence the number of towers with the required condition is 12+12+12=36.

18. The World Series Problem (January 1999, Grade 11) – In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: a) in four games? b) in five games? c) in six games? d) in seven games?

The number of ways for a team to win the series (four games) in n games is the number of ways it can win three times in n-1 games (and then win the last game). This is given by $C(n-1,3)$. The probability of any given set of outcomes for n games is $1 \div 2^n$ (since there are two equally likely outcomes for each game). So the probability that one team wins the series in n games is given by $C(n-1,3) \div 2^n$, and the probability of a win for either team is double that: $C(n-1,3) \div 2^{n-1}$. The probabilities are:

a) $C(4-1,3) \div 2^{4-1} = C(3,3) \div 2^3 = 1 \div 8 = 0.125$.
b) $C(5-1,3) \div 2^{5-1} = C(4,3) \div 2^4 = 4 \div 16 = 0.25$.
c) $C(6-1,3) \div 2^{6-1} = C(5,3) \div 2^5 = 10 \div 32 = 0.3125$.
d) $C(7-1,3) \div 2^{7-1} = C(6,3) \div 2^6 = 20 \div 64 = 0.3125$.

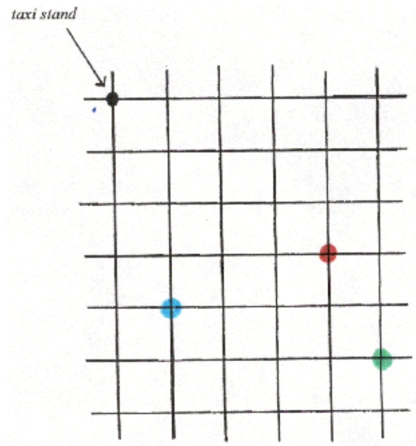
19. The Problem of Points (February 1999, Grade 11) – Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing "winner takes all." But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. They carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 francs be divided?

We can list all the circumstances where Fermat gets two points before Pascal gets three points. He can do this in two flips, three flips, or four flips. (The game cannot proceed past four flips. As soon as both players get to nine points, the next flip will produce a winner. It takes three flips for this to happen.)

a) Two flips: Fermat wins both. Probability $=1 \div 2^2 = 1 \div 4$.
b) Three flips: Fermat wins one of the first two and the last one. Probability $= C(2,1) \div 2^3 = 1 \div 4$.
c) Four flips: Fermat wins one of the first three and the last one: Probability $= C(3,1) \div 2^4 = 3 \div 16$

Probability of any of these events $= 1 \div 4 + 1 \div 4 + 3 \div 16 = 11 \div 16$. Therefore Fermat should get $100 \times 11 \div 16$ Francs ≈ 69 Francs and Pascal should get 31 Francs.

20. The Taxicab Problem (May 2002, Grade 12) – A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route. What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



Using Powell's (2003) notation to denote coordinates on the taxicab grid, (n,r) indicates a point n blocks away from the taxi stand and r blocks to the right. So the blue dot is at (5,1), the red dot is at (7,4), and the green dot is at (10,6). Taking the shortest route means going in two directions only (down and to the right). Finding the number of shortest paths from the taxi stand (0,0) to any point (n,r) involves the number of ways to select r segments of one kind of movement in a path that includes two kinds of movements; i.e. the number of shortest paths to (n,r) is $C(n,r)$. For the specific cases given above, the shortest paths are:

Blue: $C(5,1) = 5$.
Red: $C(7,4) = 35$.
Green: $C(10,6) = 210$.