Chapter 7: Block Towers: From Concrete Objects to Conceptual Imagination

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Date and Grade: March 13 & 27, April 17, 1996; Grade 8
Tasks: Binomials and towers
Participant: Stephanie
Researchers: Carolyn A. Maher and Robert Speiser

7.1 Introduction

In previous chapters, we looked at the development of various forms of reasoning in students working in a classroom in small group settings. In this chapter, we focus on an individual student – we examine Stephanie's development of combinatorial reasoning. In previous chapters, we saw how Stephanie, working with others and on her own, made sense of the towers and pizza problems. In this chapter we see how Stephanie extended that work. In her examination of patterns and symbolic representations of the coefficients in the binomial expansion using ideas from earlier explorations with towers in grades 3 through 5, she examined several fundamental recursive processes, including the addition rule in Pascal's Triangle.

This chapter centers on how children can build fundamental mathematical understanding, over time, through extended task-based explorations. They create models, invent notation, and justify, reorganize, and extend previous ideas and understandings to address new challenges. By the time of the interviews that we report here, we had been observing Stephanie for eight years. Her work in combinatorics began in grade 2, with the shirts and jeans problem (refer to Chapter 3). Even at this early stage, she would validate or reject her own ideas and the ideas of others, based on whether they made sense to her or not. Stephanie would monitor and often refer to ideas and conclusions of other group members, and would often integrate the ideas of partners in her work and discourse. This constant, extended process of evaluation and revision helped her to keep track of data, and to reconsider, strengthen, and extend her explanations (Davis, Maher, & Martino, 1992; Maher & Martino, 1991, 1992a, 1992b).

In grade three, Stephanie was introduced to investigations with block towers (see Chapter 4) that enabled her to build visual patterns of her ideas, such as the local organization within specific cases, based on ideas like "together," "separated, " "how much separated." She recorded tower arrangements first by drawing pictures of towers and placing a single letter on each cube to represent its color, and later by inventing a notation of letters to represent the colored cubes. Stephanie's working knowledge about towers, gained over long periods of time through very concrete explorations, led as, we shall see, to powerful and personally meaningful new ways to work with mathematical ideas.

7.2 Theoretical Perspectives

We believe that children come to mathematical investigations with theories that can be built upon, modified, and refined. In turn, children's theories and their ways of working with these theories help us, as researchers, to constitute our own conceptions of children's emerging work and thought, and so affect the way we build the discourse, day by day, that we will share with them. In the task-based interviews that we report, we, too, will seek to build a theory. Our emphasis on building theory informs directly how we structure research interviews. Initially, one interviewer will engage the child in a specific exploration, seeking to estimate the working theory that might guide the child's thinking. Later, in the same interview or in a subsequent follow-up interview, key ideas noted so far will pursued primarily by the child, who initiates, and then increasingly directs, the discourse. In such interviews, we frequently begin with very concrete discussions, followed by what might be called a "teaching phase" intended to investigate deeper connections. In such interviews, children will sometimes make powerful connections early, and so break the flow we might naively have imagined. We have come to view such "unique outcomes" as potential opportunities to gain important insights from the children that we study. Therefore, when a child's connection appears to break the flow, the interviewer, on principle, will invite more detailed explanation.

In *Mindstorms* (1980), Seymour Papert reflects on how he built his personal mathematical understanding — an understanding that inspired his later work — based on his personal experience, as a young child, playing with gears. In a similar way, some of Stephanie's key mathematical understandings can be traced to her activities, in the early grades, when she used block towers to investigate conceptually important counting problems.

The specific arguments that Stephanie investigates below were first developed and explained in Speiser's paper (1996), where block towers underpin a concrete microworld for productive exploration. These arguments, shaped specifically within the given microworld, were triggered by the early "Gang of Four" investigations (Maher & Martino, 1996a, 1996b and Chapter 4), which first describe the reasoning and argument that enabled Stephanie and three other children, at age 9, to discover the *idea* of mathematical proof, as they built and then debated strategies for counting block towers. Building from this work with towers (and inspired by the young Papert) we seek precise, particular descriptions (1) of how Stephanie actually does strong mathematics, based on towers, and (2) of what, specifically, might constitute its strength.

7.3 Setting

Stephanie participated in the longitudinal study starting in first grade. Stephanie and her classmates were challenged in their mathematics classrooms to build solutions to problems and construct models of their solutions. This setting, which for Stephanie continued to grade 7, encouraged differences in thinking that were discussed and negotiated. In fall 1995, Stephanie moved to another community and transferred to a parochial girl's school. Her mathematics program for grade eight was a conventional algebra course. Stephanie continued to participate in the longitudinal study through a series of individual task-based interviews. A subset of these interviews provides the data for this chapter.

7.4 Guiding Questions

The following questions guide our analysis in order to consider, systematically the ways in which Stephanie's past experience is drawn upon: (1) How does Stephanie work with towers in building images and understandings for higher mathematical ideas? (2) What is the role of past experience in building new ideas? (3) How are her ideas modified, extended, and refined over time?

Data come from two of eight individual task-based interviews of Stephanie. The interviews were videotaped with two cameras, positioned to capture in detail what was said, written and built, and to include less tangible data such as tone of voice, speech tempo, and where people look while they converse and work. Transcripts and analyses of the interviews were made and verified by a team that included several graduate students in addition to both authors. Stephanie's written work prepared the interview, and several observers' notes, provide further data. The teaching experiment was conducted over a six-month interval (November 8, 1995 to May 1, 1996). Each interview, approximately one and one-half hours in length, would typically begin with inquiries about the mathematics that Stephanie currently studied in eighth-grade algebra, both to open opportunities to talk about that mathematics, and to explore her thinking about fundamental mathematical ideas.

7.5 Results

To introduce each data segment, we provide a brief discussion of the mathematics Stephanie was invited to explore. On this basis, we can more clearly understand each segment as a momentary snapshot of Stephanie's emerging understanding.

The correspondence between binomials and towers. During the March 13, 1996 interview, Stephanie, unprompted, made a connection to towers, by examining her symbolic representation of the expansion of $(a + b)^2$ and $(a + b)^3$.

STEPHANIE:	So there's *a* cubed [a^3].
RESEARCHER:	That's 1.
STEPHANIE:	And there's three *a* squared *b* [$3a^2b$] and there's three *a b* squared [$3ab^2$] and there's *b* cubed [b^3]. [Interviewer writes 1 3 3 1 under 1 2 1 as Stephanie speaks.] Isn't that the same thing?
RESEARCHER:	What do you mean?
STEPHANIE:	As the towers.
RESEARCHER:	Why?
STEPHANIE:	It just is.

Stephanie asserts (in her own way) that each 3-high tower gives a non-commutative monomial of degree 3 in *a* and *b*, and she has indicated that these non-commutative monomials, indexed by the corresponding towers, collect to give the coefficients for the commutative monomials that appear in her expansions of $(a + b)^2$ and $(a + b)^3$. Our interpretation, therefore, is that Stephanie visualizes towers (referring to mental models—she does not have plastic cubes at this point) to help her organize the terms that she collects. We believe that Stephanie reasons about polynomials based on her mental images of towers.

Working at home before the interview, Stephanie had written out the first six powers of the binomial a + b, and brought her written calculations to the interview. The interviewer covered Stephanie's paper, guessed the coefficients for the sixth-power expansion, and wrote down the terms in full. Her coefficients were the same as Stephanie's, although one monomial was slightly different. Several minutes further in the conversation, Stephanie gives further evidence, that she proceeds by visualizing towers and then reasoning based on her mental images.

RESEARCHER:	So you have two factors of *a*. Right?
STEPHANIE:	Um hm.
RESEARCHER:	You have one of those. One thing with two factors of *a*. One thing with two *a*'s in it.
STEPHANIE:	Um hm.
RESEARCHER:	I don't want to think of *a*'s. I want to think of red.
STEPHANIE:	Okay [laughing].
RESEARCHER:	Can you switch that a minute?
STEPHANIE:	Yeah.
RESEARCHER:	So now I have one thing with two reds. What thing can I be thinking of with two reds?
STEPHANIE:	That's a tower that's two high.
RESEARCHER:	Okay. And here I'm talking about two things.
STEPHANIE:	Um hm.
RESEARCHER:	One is.
STEPHANIE:	Red and...
RESEARCHER:	one is...
STEPHANIE:	...one is yellow.
RESEARCHER:	Is that possible in two high?
STEPHANIE:	Yeah.
RESEARCHER:	Having one red and one yellow? There are two of them?
STEPHANIE:	Yeah.

RESEARCHER:	Which two?
STEPHANIE:	'Cause the one is the red could be on the top or the bottom, with the yellow the same thing.
RESEARCHER:	What about *b* squared?
STEPHANIE:	Um. Two yellow.

In a March 27, 1996 interview, Stephanie is invited to explain to a second interviewer (unfamiliar with her recent work) what had happened in the March 13 interview described above. Here Stephanie begins with towers, then reviews the binomial coefficient notation C(n, r), working through a sequence of examples with increasing n. Stephanie remarks that "r is a variable," which she understands can range from *0* to n. This observation shifts the level of abstraction upward from specific towers (as above) to patterns of formal symbols as in Pascal's Triangle. Hence, at this point, n, the height, and r, the number of red blocks for given n, will both vary. This richer context triggers, with encouragement from Interviewer 1, a confident, detailed and carefully presented recapitulation by Stephanie of the recursive construction of the towers of height n from the towers of height $n-1$, as it had been introduced by classmate Milin in grade 4 and revisited ;min grade five (see Chapters 4 and 5).

During a previous interview, on March 13, Stephanie also referred to Pascal's Triangle, in particular to its addition rule, to make similar predictions, but she had done so in a conceptually quite different domain: to predict, in effect, the numbers of n-tall towers in each given case (of r red blocks, say, for given r) for new values of the height. Stephanie's choice to center, in the present interview, directly on binomials strongly suggests that Stephanie now grasps the isomorphism between Pascal's Triangle, which she had built, at first, to summarize her *towers* cases, and the array of coefficients for her polynomial expansions of the powers $(a + b)^n$, for variable n.

On this basis, further interviews were planned, with towers available to serve as concrete anchors to establish formal facts about the C(n, r), viewed either formally as binomial coefficients or as counts of combinations, or, more concretely, as the numbers of specific kinds of towers.

Fermat's recursion. One goal for the March 27th exploration was to offer Stephanie the tools she'd need to construct a formula, originally due to Fermat (Weil, 1984) that expresses the relationship between two successive binomial coefficients. In symbols, here is Fermat's formula.

$$C(n,r+1)=\left(\frac{n-r}{r+1}\right)\cdot C(n,r) \tag{1}$$

This equation, applied repeatedly beginning with the simple case $r = 0$, leads directly to the standard formula for $C(n, r)$.

To make sense of this formula, it seems especially helpful in this setting to interpret equation (1) in terms of towers. For concreteness, take red and yellow for the colors of the blocks available. On the right side, $C(n, r)$ counts the towers of height n that have exactly r red blocks, hence $n - r$ yellow blocks. Call these the *original towers*. On the left side, $C(n, r + 1)$ counts the towers of height n with exactly $r + 1$ red blocks. Call these the *new towers*. In concrete terms, equation (1) tells us that the number of new towers can be found by multiplying the number of old towers by the number of yellow blocks in each, and then dividing by the number of red blocks in a new tower.

In the data below, the interviewers will fix n, the height of a tower, and then vary r, beginning either with $r = 0$, or $r = 1$, for which $C(n, r)$, is either known to Stephanie or easily determined by inspection. For each r, the interviewers will then invite Stephanie to construct new towers from a given set of original towers, and explore with her what she has found. The construction process Stephanie explores will work for any height n and any $r < n$.

For concreteness, we explain this process when $n = 4$ and $r = 1$. In this case, we have four original towers, each with a red block in one of four available positions. From each given original tower, we can build new towers by replacing one of its three yellow blocks with a red block. For each of the four original towers, we can therefore build three new ones. Working in this way (we'll say *by day*) we obtain four groups of three new towers. The 12 towers constructed in this way clearly include each possible new tower. For example, consider Figure 7-1 as an example: Working by day, begin with one original tower four blocks high with one red block (shaded). We obtain three new towers, each with two red blocks, by replacing in turn each of the three yellow blocks in the original tower with a red block.

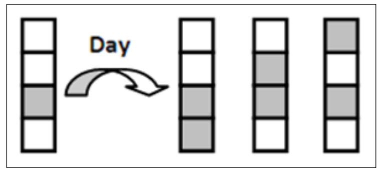

Figure 7-1. Working by day: replace each yellow block (unshaded) by a red block (shaded)

The total we have just obtained instantiates the product $(n - r)\cdot C(n, r)$ on the right side of equation (1), but — this is a key point — the new towers we just built are not distinct. In fact, each tower appears exactly *twice* among the 12, as the denominator $r + 1$ predicts. To understand how duplicates emerge, consider a particular new tower. This tower has exactly *two* red blocks. Each of these two red blocks can be replaced (working, we shall say, *by night*) with a yellow block, producing one of two original towers. This construction, which reverses what we did by day, shows that each new tower will appear exactly twice among the 12 we had constructed. In particular, there will be exactly 6 towers of height 4 that have exactly 2 red blocks. Because towers correspond to combinations, we have used the known result $C(4, 1) = 4$ to show that we have $C(4, 2) = 6$. For example, consider Figure 7-2. By night, begin with a new tower four blocks high ($n = 4$) with two red blocks. We obtain two original towers, each with one red block, by replacing one of the two red blocks in the given new tower with a yellow block.

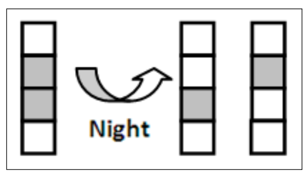

Figure 7-2. By night, replace each of the red blocks (shaded) by a yellow block (unshaded)

Stephanie began to explore the construction shown in Figure 7-1 during the March 27 interview, first with three-tall towers. Her blocks were blue and green. Continuing to four-tall towers, she next built the four towers with exactly one green block, and guessed initially (but incorrectly, perhaps based on her experience with three-tall towers) that from each such original four-tall tower she could obtain two new ones.

RESEARCHER:	I wonder why you get two of them.
STEPHANIE:	I don't know. Maybe cause it's bigger.
RESEARCHER:	What would that have to do with it?
STEPHANIE:	I don't cause you have more room to build on.
RESEARCHER:	Tell me, can you explain to me?
STEPHANIE:	Oh, well, maybe it's because like you already have one [green block] that's taking up space, so you only have three places to move it.
RESEARCHER:	I gotcha, okay.
INTERVIEWER 2:	Okay.
RESEARCHER:	So what would you predict if you were building towers five high?
STEPHANIE:	You'd have four.

Here we see Stephanie revise and then explain her observations, starting from a set of four-tall towers that she had physically built. On this basis, she extends her observations to a set of five-tall towers she has just imagined. So far, she knows that duplicates appear in the construction she discusses, but has not yet explored in detail how or why they do.

Revisiting the same construction in the next interview session (April 17, 1996), Stephanie considers duplicates directly. This time (as in the earlier examples) her colors will be red and yellow, and the variable r will count the red blocks in a tower. After reviewing, for four-tall towers, the construction of new towers from original towers, with r

= 0, 1, 2 and 3 in succession, the researcher invites Stephanie to predict how many the duplicates she would expect for towers of height 5, and helps her build the towers that she needed as they proceed. They begin with a tower with five yellow blocks (the case $r = 0$).

STEPHANIE: The first one's one.
RESEARCHER: There's one of those
STEPHANIE: times five.
RESEARCHER: Why five?
STEPHANIE: 'Cause there's five positions.
RESEARCHER: Okay.
STEPHANIE: Divided by one, 'cause they come in groups of one.
RESEARCHER: Um, hm.
STEPHANIE: Five.
RESEARCHER: Okay. So that's five things taken one at a time.
STEPHANIE: Yes. The second one
RESEARCHER: Why don't you write that down? Five things taken, equals five things taken one at a time. [Stephanie writes.]
STEPHANIE: Okay. For the second one, um, there's four spaces. But there's—out of five—so its five times four and they'll come up in groups of— I don't know, um, that's what we don't know though.
RESEARCHER: All right. So. Let's—can we make these five? Just, here.
STEPHANIE: Well, maybe they might come in groups of two?
RESEARCHER: One. Let's think about at least one of these.
STEPHANIE: They might come in groups of two, I guess.

Here, just as they begin to build the five five-tall towers with one red block, Stephanie repeats, it seems, the mistaken guess that she had made earlier for four-tall towers. At this point, the interviewer arranges the five towers they have built in front of Stephanie and offers Stephanie the tower with its one red block in the top position.

RESEARCHER: Okay. So what you're saying here — move some of this aside — um, okay. Let's think of that one.
STEPHANIE: Okay.
RESEARCHER: There are five.
STEPHANIE: [Builds a tower with a second red block just below the first.] You have one like that. [She builds another tower with a second red block two spaces below the first.] One like that.
RESEARCHER: Well, can you predict before you do it?
STEPHANIE: Yeah, there's going to be four from each.
RESEARCHER: Four from each.
STEPHANIE: Yeah.
RESEARCHER: Okay. So—and what's the each? How many make up each?
STEPHANIE: How, wh, what do you mean?
RESEARCHER: You're saying, it's four from this.
STEPHANIE: Yeah, four from
RESEARCHER: What does-
STEPHANIE: One.
RESEARCHER: -each mean in this case?
STEPHANIE: Oh! Like there's going to be four from this one. Four from that one. Four from that one. Four from that one. Four from that one.
RESEARCHER: Okay. So how many eaches?
STEPHANIE: There's five.
RESEARCHER: Five eaches. Okay.
STEPHANIE: Yeah.
RESEARCHER: All right. So that, you say, five times four.

STEPHANIE:	Yes. I have that. I just don't know what the-
RESEARCHER:	Right.
STEPHANIE:	-bottom part—it
RESEARCHER:	So—and by the groups, you mean. The groupings you mean.
STEPHANIE:	Groups like—one after we've put them all out. Like how many groups, they're going to come in-
RESEARCHER:	I don't know. I'm
STEPHANIE:	duplicates?
RESEARCHER:	I'm wondering. When you say you divide by
STEPHANIE:	Oh! 'Cause that's the number of duplicates—that there are.

Again, working by day, with towers on the table, Stephanie corrects her guess, but then—a new step—tries to go further. She has just built 20 towers in five sets of four. Now she proposes to *restructure* her set of 20 towers into a groups of duplicates, or at least to find the number of such groups. In effect, she has proposed the key step by herself: to find how many duplicates each new tower has.

RESEARCHER:	But how do you know beforehand? Do you think there's a way?
STEPHANIE:	[Building towers.] Oops.
RESEARCHER:	So if this, um, is going to be a pattern to this—the five times four—what do you think you would divide by?
STEPHANIE:	Five times four—what do you think I'd—um—maybe two.

Working backward from the known entry, 10, in Pascal's Triangle, Stephanie confirms that she indeed will need to divide 20 by two. The explorations then continue with ten towers of height five with two red blocks. Working by day, Stephanie predicts that 30 new towers that can be built beginning with her ten originals. This time each new tower will have three duplicates.

STEPHANIE:	Ten. So it would be ten times three and you divide by three. [Writes as she speaks.]
RESEARCHER:	And it worked?
STEPHANIE:	Yeah. And the next one, there is two spaces to put it and you have ten. So there's ten times two, and you divide by two? [Continues writing.] And the last one — there's one space to put it — it's five times one divided by five equals one.

In this exploration, in effect, Stephanie has explained how the corresponding row (1, 5, 10, 10, 5, 1) of Pascal's Triangle emerges numerically from the pattern of equation (1), which she *has not yet seen*. At each step, she connects the product $(n - r)\cdot C(n, r)$ directly to the operation of replacing one yellow block with a red block.

We note, however, that Stephanie has not yet identified the denominator, $r + 1$, with the number of red blocks in a new tower. Instead, she seems to follow a numerical pattern that she has observed empirically. At this point, Stephanie does not yet seem able to explain why the number of duplicates that she observes must *necessarily* be $r + 1$. Nonetheless, we remain astonished, after twelve years, by the depth and strength of the connections Stephanie has made, based on her familiarity with towers.

The addition rule in Pascal's Triangle. By March 1996, as noted above, Stephanie already knew the additive pattern that relates successive rows of Pascal's Triangle. In symbols, this *addition rule* can be expressed as follows.

$$C(n+1,r) = C(n,r-1) + C(n,r) \quad (2)$$

According to this formula, each row of Pascal's Triangle can be computed from the row before it, by adding each pair of successive entries in the row above. To connect this formula to combinations, and in this way make sense of it, we will read each term as a count of towers. Specifically, the first term on the right counts the towers of height n that have exactly $r - 1$ red blocks, while the second term counts towers of the same height, but with one additional red block. So interpreted, the right side of equation (2) at least *suggests* that every tower of height $n + 1$ that has exactly r red blocks *can be constructed* from suitable shorter towers of height n, either by placing a red block on top a tower of height n with $r - 1$ red blocks, or by placing a yellow block on top of a tower of height n that with exactly r red blocks. A special case is shown in Figure 7-3: In the top row, we begin with two sets of towers of height three ($n = 3$): one tower with no red block ($r = 0$), and three towers with one red block ($r = 1$, shaded). To accomplish the recursion, attach a red block (R) on top of the single tower in the first set, and a yellow block (Y) on top of each tower in the second set, to produce four towers of height 4 with one red block.

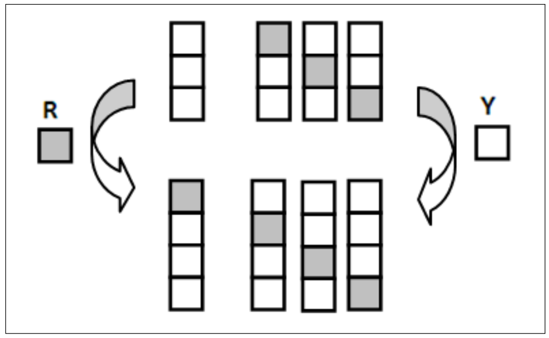

Figure 7-3. A specific example of the addition rule for Pascal's Triangle

Again we work both day and night. By day, attaching blocks as shown in Figure 7-3, it is not difficult to see that the resulting new towers of height $n + 1$ must be distinct. Then (by night), if we remove the top block of each possible tower of height $n + 1$ that has exactly r red blocks, it's clear that all such towers have been counted on the right side of equation (2).

In the data segment soon to follow (later in the April 17, 1997 session) Interviewer 1, drawing several row of Pascal's Triangle, writes down the numbers 1 and 3 that correspond to the towers shown in the top row of Figure 7-3. She is just about to write the number 4 below them, and then draw two diagonal lines, to associate the numbers 1 and 3 to the 4.

RESEARCHER:	Okay. Um. Let's explore, um — which one should we explore? [Draws lines as above.] Let's do this one.
STEPHANIE:	Um, hm.
RESEARCHER:	Do you know what this one means? If you had to build this one, what would that tower look like?
STEPHANIE:	That one?
RESEARCHER:	What would that one look like? What would those two look like?
STEPHANIE:	[A pause, while Stephanie builds towers.] I think that one would be like this. [Stephanie has built the tower of height 3 with all yellow blocks, and she indicates the one that Interviewer 1 has drawn] and that one.
RESEARCHER:	Three high, no red.
STEPHANIE:	Like this. [She has just built the first two towers of height three with one red, as in Figure 7-3.]
RESEARCHER:	Okay. Three high, exactly one red.
STEPHANIE:	Yes.
RESEARCHER:	Okay.
STEPHANIE:	Oh! Wait! [Builds the remaining tower.]
RESEARCHER:	Okay. Makes you dizzy after a while, doesn't it? 'Cause I think I see exactly one also. Even when you make it, I just believe you're gonna do it. Okay. Now. When we're doing this [points to the 1, the 3, and the 4 that she has written].
STEPHANIE:	Um, hm.
RESEARCHER:	What's different about these and this tower here [taps the number 4 in Pascal's Triangle] that I call four? There.
STEPHANIE:	Well — it's four high.

For a few lines, Interviewer 1 and Stephanie review the towers of height three that Stephanie has built and has physically in front of her. They easily agree that the number 4 that Interviewer 1 has written beneath the entries 1 and 3 should count the towers 4 blocks high that have exactly one red block. These towers have not yet been built, and *they will not be built* in the conversation that will follow.

RESEARCHER:	I want to know from here
STEPHANIE:	Uh, hm.
RESEARCHER:	what you do to these [the towers of height 3]-
STEPHANIE:	Well.
RESEARCHER:	-to get me, to get me-
STEPHANIE:	Well, I'd build them higher.
RESEARCHER:	Well, don't do it yet. Just think about it for a minute. Remember what they're going to look like.
STEPHANIE:	Yeah.
RESEARCHER:	There's going to be exactly one red.
STEPHANIE:	This would go here. [She moves the all-yellow tower of height 3] and there would be red.
RESEARCHER:	No. No. We start with these [points to the number 4 again]. I don't want you to touch these [indicates the towers of height 3]. I want you to tell me what you're gonna do to these so that when you're all done-
STEPHANIE:	Um, hm.
RESEARCHER:	-you end up with exactly one red. But you've got to make them all four tall.

This point is delicate. Stephanie knows (empirical experience!) the four-high towers with exactly one red block, and she can easily imagine them. But does she understand, working by day, how those towers *can be built* from the four towers of height 3 she has in front of her?

STEPHANIE:	I'm going to put a yellow here [points to the first tower on the right in the top line of Figure 7-3],
RESEARCHER:	Okay.
STEPHANIE:	I'm going to put a yellow here [points to the next tower],
RESEARCHER:	Right.
STEPHANIE:	I'm going to put a yellow there [points to the third tower in the same group] and I'm gonna put a red there [points to the all-yellow tower].
RESEARCHER:	Okay. So how many ways—how many do you end up with?
STEPHANIE:	Four.
RESEARCHER:	Four. So from the one three tall with no reds
STEPHANIE:	Um, hm.
RESEARCHER:	And the three three-tall with one red, right?
STEPHANIE:	Yes.
RESEARCHER:	You end up four four-tall with one red.
STEPHANIE:	Yes.
RESEARCHER:	Isn't that neat?
STEPHANIE:	Yeah.

In this and later interviews, Stephanie first masters this way of working in continued conversations with Interviewer 1, but she then goes on, in later sessions with her peers, to teach the line of reasoning she begins exploring here to others.

A conceptual reflection. In these data, we see Stephanie refining and revising new ideas that she has built from raw materials she draws from prior experience with towers and combinations. This prior experience includes a variety of proofs (first by cases, later by induction), expressed concretely with block towers and more formally through language and notation that she and peers have personally developed and refined throughout their long collaboration.

This process of revision and refinement, that we emphasize throughout, might be most clearly visible across the data we see here as a progressive movement from sets of towers that Stephanie built physically (See Figure 7-3), toward sets of towers that Stephanie comes to *imagine*. These imagined towers (such as the final set of four above) are not simply visualized as static images from prior tasks; indeed, the new towers have been constructed based on *new* conceptual ideas that Stephanie has begun to build, *in real time, as the interviews proceeded*.

Figure 7-3. Stephanie's tower exploration, grade 8

7.4 Discussion

In an earlier paper, based on a just fraction of the data we considered here, we used the metaphor of text to state the following conclusions (Maher & Speiser, 1997b, p. 131):

> Images, patterns, and relationships have become mathematical objects that Stephanie sees and works with mentally to build abstractions. Our conversations with her elicited both spoken and written texts. These texts, together with our interpretations, anchor an analytic narrative of the development of certain mathematical ideas. Such texts (which we propose to view as work in progress) extend through time and serve as records of particular events upon which later texts can comment. Further, they can serve as raw material from which new texts can be composed.
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> We revise our texts, and so does Stephanie, as our experiments proceed through detailed interactions with each other. Hence, as Stephanie's developing judgment enters the discussion, her presentations offer raw materials that help to focus and direct the researchers' later task designs and explorations. Our agenda for the interviews, seen as an emerging text, continues to be rewritten, reconsidered, and revised, often in direct response to goals that Stephanie pursues.

After the interviews we have considered here, events that neither Stephanie, her peers, nor the researchers could foresee in 1996 would offer opportunities for everyone involved to deepen, reconsider, refine, and extend their previous perspectives and conclusions. In the discussion we have just presented, perhaps most striking is the heightened prominence we see of personal, conceptual imagination to address new problems and, in the process, to give form to new and powerful ideas.

In the next chapters, we follow other students from the longitudinal study who also build on previous explorations to make sense of Pascal's Triangle and rules for its generation.