# Chapter 2: Methodology

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## *2.1 Introduction*

In this chapter, we discuss how data was collected and analyzed, and we briefly describe some results, which will be more fully explored in later chapters. We summarize student work on fundamental problems and note how this work led to exceptional growth in the students' mathematical understanding.

Researchers (professors at the Rutgers University Graduate School of Education and their students) conducted all problem-solving sessions with the students; the sessions were always videotaped with one or more cameras. Researchers observed, described, and coded the videotape data, and they kept written and electronic files of the emerging theoretic, analytic, and interpretative ideas about the students' mathematical behaviors. Researchers paid careful attention to children's use of inscriptions, the connections they made between and among codes, and their emerging and extended ideas and ways of reasoning. Critical events in children's reasoning were flagged and transcribed and transcripts were coded according to the research questions. The connected series of events that formed a trace led to the emergence of a narrative (Maher & Martino, 1996a; Powell, Francisco & Maher, 2003).

The videotapes, researcher notes, and student notes did not capture every interaction or every case of student learning. Some students sat silently during discussions; but they had quietly absorbed a problem or quietly developed a solution that came to light some time later in a different situation. Therefore, although we can make inferences about what is observed, we cannot assume that a student who is quiet or who seems to say the wrong things does not understand.

By videotaping children as they worked together on mathematical tasks over long periods of time, we were able to trace the origin and development of their mathematical ideas. We observed what children said to one another and showed to one another. We used videotapes and transcripts to study the meanings that children gave to mathematical situations and to note the different representations they made public. A detailed analysis of data made it possible to trace the origin and evolution of children's arguments. Our data indicate how children expressed their ideas through spoken and written language, through the physical models they built, through the drawings and diagrams they made, and through the mathematical notations they invented.

## *2.2 Theoretical Perspectives*

Guiding our work is the view that children come to mathematical investigations with theories they can modify and refine. We observe them do so in settings that combine personal exploration and suitable social interaction. The theories we consider can include criteria to decide (1) what, at some given moment, needs to be investigated, (2) how to conduct such an investigation, (3) what key features need to be explored in detail, (4) when useful progress has been made, and, given such progress, (5) if further investigation might be needed. We have found that theories of this kind often empower striking and effective ways for children to work conceptually with mathematical ideas, often using concrete objects as specific anchors for their thinking.

## *2.3 Selected Problems*

Mathematics arose from the need to count, measure, and calculate, but the discipline evolved to include abstraction, logical reasoning, and the search for and analysis of patterns. Good mathematical problems are therefore those which give rise to the need for abstraction, systematization, and pattern-recognition. A focus of the study was therefore to select problems that would give rise to these needs.

Another focus of the longitudinal study was on doing problems that were not part of the regular curriculum, because it was important for the students to come to the problems fresh, without pre-taught algorithms. A major strand of the longitudinal study therefore consisted of problems in combinatorics, because in working on these problems, students can find the need to organize their work systematically, look for patterns, and generalize their findings; also, counting problems were at the time outside the regular elementary school curriculum and therefore unfamiliar to students. In addition, these problems lend themselves to the use of multiple personal representations that can be shared. Freudenthal (1991) cites the study of combinatorics as “a most important matter for reinvention” (p. 53), specifically because combinatorics can be learned through paradigmatic examples and because problems in combinatorics give rise to the need for convincing proof, including mathematical induction.

Another purpose of the longitudinal study was to provide an environment in which certain sociomathematical norms could be established to elicit in children sense making, argumentation and justification in mathematics. As Yackel and Cobb (1996) and Cobb, Wood, Yackel and McNeal (1992) suggest, an appropriate social context must be created to encourage students to try to convince others of the truth of the mathematical ideas that they build. The longitudinal study was structured to investigate and track the nature of the schemes that students developed and the methods that the students used to build and retrieve representations to solve mathematical tasks. In addition, the study attempted to trace how students shared ideas and how these ideas were adapted and assimilated by other students.

Appendix A describes all the combinatorics problems that students worked on over the years. We summarize here some example problems, along with brief accounts of strategies and representations used by students and forms of reasoning that developed.

### 2.3.1 Shirts and Jeans

Students worked on the shirts and jeans problem at the end of second grade and again at the beginning of third grade (1989 and 1990).

> *Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?*

During second grade, most students drew pictures of outfits; some drew lines between shirts and jeans, and others made lists of outfits. Notational choices influenced the way they reasoned about the data. For example, Stephanie used “blue-white” to stand for the white shirt/blue jeans outfit, and also for the blue shirt/white jeans outfit. Contextual issues also played a role in the problem solving. For example, Dana discarded the white jeans/yellow shirt outfit on grounds that the resulting outfit didn’t match and was thus not fashionable. That different students got different answers was not problematic for the children; in second grade, students seemed comfortable with the notion that answers varied between three and seven outfits. They willingly shared their interpretations and strategies and talked to each other about their findings. In third grade, when the children were again presented with this problem, they did not remember how they had solved the problem earlier; nor did they remember their earlier answers. Of particular interest is that evidence of further elaboration of earlier strategies emerged. Students used and built on strategies of their second grade partners. For example, Stephanie indicated different outfits by drawing lines between drawings of shirts and jeans, as Dana had done in second grade.

By third grade, techniques for checking and for keeping track, such as controlling for variables, were complete. Earlier ideas and strategies were refined to produce complete, elegant solutions rather quickly. Students built on their heuristics to solve more complex extensions of the problem to include belts and hats as parts of outfits.

What was especially significant for the researchers was the evidence of how students built on earlier ideas and, without intervention or approval from researchers, continued their problem solving, driven by earlier heuristics and sense making to produce correct solutions that they could justify.

### 2.3.2 Towers

Early in third grade (1990), students were given the four-tall tower problem for the first time:

> *Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.*

The definition of a tower is: an ordered sequence of Unifix cubes, snapped together. Each cube can also be called a *block*. Each tower has a *bottom* and *top*. The *height* of a tower is the number of its cubes. We say two towers are the *same* if their colors match, block by block, from top to bottom. Unifix cubes are interlocking cubes that come in various colors (typically blue, red, yellow, white, and green).

In fourth grade (1991), students worked on the five-tall tower problem. Then in fifth grade (1992), they revisited the four-tall version. In tenth and eleventh grades, they were asked to provide a justification for the *n*-tall tower problem. Students discussed variations and generalizations of the solution and they used their organization of towers by cases and knowledge of the binomial expansion to build an understanding of how Pascal's Triangle grows.

Their work on the towers problems also illustrates how their representations changed over the years. At first, they used Unifix cubes to build towers. Eventually, they turned to drawings and codes, for example using letters *R* and *Y* to mean red and yellow cubes. In some cases, a more general code emerged; some students would use *X* and *O* or 0 and 1 to indicate any two colors. More details on these emerging strategies are given in Sections 2 and 3.

### 2.3.3 Pizzas

In order to introduce a variation of the tower problem and to investigate how students reasoned with an isomorphic problem, the researchers introduced the set of pizza problems. When the students were first given the problem in fifth grade, they interpreted the task as allowing different toppings on each half of the pizza, an alternative that they knew that was available in some pizza restaurants. In response to their interest in counting the varieties allowing toppings on half a pizza, the researcher asked them to solve it with only two toppings available. This pizza with halves problem is:

> *Kenilworth Pizza has asked up to help them design a form to keep track of certain pizza sales. Their standard plain pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pie or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole cheese pizza or half a cheese pizza? List all possibilities. Show your plan for determining the choices. Convince us that you have accounted for all possibilities and there could be no more.*

The strategy that the students developed for the solution established the heuristic that was applied later when there were five toppings available, again, allowing some or no toppings on half the pizza. The final problem, the five-topping pizza problem, proved a trivial special case for the pizza with halves problem that they first solved successfully.

> *The local pizza shop offers a plain cheese pizza. On this cheese pizza, you can place up to five different toppings. How many pizzas is it possible to make?*

*Pizza with Halves* was the first of several variations of the pizza problem that the students worked on over the years. It illustrates a basic philosophy of the study – we did not start students off with easier problems and then progress to the more difficult ones. Instead, students began with the more difficult versions of the problems, which required them to tackle several challenges at once – organization (making sure no pizzas were repeated and none were omitted), notation (how to distinguish between pepperoni and peppers, for example), and forming a valid argument – how to convince the researchers (and themselves) that they had the right answer.

Looking at students' answers to the pizza problems over the years, we see growth in organization and in representations. At first, students drew fairly accurate renditions of pizzas; they drew circles to indicate pizzas, and inside those circles were wavy lines to indicate sausages and smaller circles to indicate pepperoni, for example. When they had to answer a question involving half pizzas, they drew lines down the middle of their pizza circles to show both halves, and they listed all the pizzas using full words (for example, "whole plain, half sausage half plain"). Eventually, they turned to codes, starting with single letters or combinations of letters (to distinguish between peppers and pepperoni, for example), and then moving to more abstract symbols such as 0's and 1's. These representations and organizational strategies are discussed more fully in Chapter 6.

In eleventh grade, some students investigated Pascal's Triangle and Pascal's Identity (the addition rule for Pascal's Triangle). Using the metaphor of the pizza problem, they explained how the triangle grows by explaining how the number of possible pizzas grows as new toppings become available. In an extraordinary session lasting over two hours one evening in 1999, students generated a slightly nonstandard but mathematically correct equation for Pascal's Identity using standard combinatorial notation:

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

A detailed description of the students' work on Pascal's Identity is given in Chapter 12.

### 2.3.4 Taxicab

The group that had generated Pascal's Identity was introduced to the Taxicab Problem in twelfth grade (2000):

> *All trips originate at the taxi stand, in the upper-left corner of a grid. The problem is to find the shortest route to three specific points on the grid and to determine the number of shortest routes to each point.*

Their work on this problem is discussed in greater detail in Chapter 13. It is interesting to note that by this time, the students, without prompting, solved the general problem, in addition to answering the specific questions. For any point on the grid, they showed why the general answer was correct, and they demonstrated the connection to isomorphic problems (towers and pizzas) and to the binomial expansion. What is also interesting from this session is that the students took on the roles of eliciting justifications from each other. Their pursuit of explanations that made sense and that connected to earlier tasks was quite remarkable.

## *2.4 Concluding Remarks*

The purpose of the longitudinal study was not to teach the students particular topics in combinatorics or other areas of mathematics. Instead, the aim was to establish a culture where the correctness of an answer came from the sense making of the students, rather than from the authority of the researcher. We asked students questions about what was convincing, what made sense, and how they developed their answers. In justifying their answers, students usually exceeded our expectations. We were impressed by the seriousness with which students approached the problems and the collegiality of their work, as well as by the forms of reasoning they developed. In the early years of the study, children began to use inductive reasoning, to organize work by cases, and to think about justification through contradiction. By middle school, these forms of reasoning were more sharply defined, and other forms of reasoning emerged, such as controlling for variables. In high school, students began the process of building isomorphisms, using their own notation as well as standard notation to describe how some problems were related to each other and ultimately to Pascal's Triangle.

In the following chapters, we provide details on the specific problems, the specific strategies and representations used by the students, and the specific results they generated. In the next chapter, we discuss the students' earliest work on combinatorics problems, the second- and third-grade work on Shirts and Jeans.