Chapter 17: Closing Observations

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In the previous 16 chapters, we have witnessed ordinary students develop extraordinary mathematical ideas, forms of reasoning, and heuristics. Extraordinary are these students' accomplishments since their mathematical behaviors emerged not from quickly parroting rules and formulae but rather from deliberately engaging their own discursive efforts. As Speiser (Chapter 7, this volume) notes, these students built fundamental mathematical understanding, over time, through extended task-based explorations. They created models, invented notation, and justified, reorganized, and extended previous ideas and understandings to address new challenges. That is, they performed mathematics: created mathematical ideas and reasoned mathematically. These behaviors—ideating and reasoning—are fundamental human activities and *how* they occur in the realm of mathematics, specifically elementary combinatorics, is what this book contributes.

Internationally, a community of mathematics education researchers has recognized this *how* question as substantially important. In January 1983, David H. Wheeler (1925-2000), the founding editor of the international journal, *For the Learning of Mathematics*, sent a letter to 60 or so mathematics educators inviting them to engage a daunting task: "suggest research problems whose solution would make a substantial contribution to mathematics education" (Wheeler, 1984, p. 40). The varied and thought-provoking responses of more than 15 educators were published, some in each of the three issues of the fourth volume of the journal. On Wheeler's mind was the famous example of the 23 problems from various branches of mathematics that David Hilbert (1862-1943) announced in an address delivered to the Second International Congress of Mathematicians in 1900 at Paris (p. 40) and predicted that "from the discussion of which an advancement of science may be expected" (Hilbert, 1900, p. 5).[1] Of all the published responses to Wheeler's challenge, Tall (1984) offered the briefest list of what he considered to be "the central questions": (1) *how do we do mathematics?* and (2) *how do we develop new mathematical ideas?* (p. 25, emphasis added).

Hilbert's 23 problems contributed to more than a century of vigorous, fruitful research activity in physics and mathematics.[2] Similarly, considered responses to Tall's two questions require substantial research efforts in different environments over extended periods of time. For researchers in mathematics education to entertain these questions, we must find ways to observe what learners do as they do mathematics as well as to describe and analyze how they develop their mathematical ideas.

It bears noting that eight years after Tall issued his central questions, Davis (1992) similarly challenged mathematics education researchers to study the emergence among learners of what lies at the core of mathematics: *mathematical ideas*. Expanding on Tall's second question, Davis noted that "very little research in mathematics education has focused on the actual ideas in students' minds or on how well teachers are able to identify these ideas, interact with them, and help students improve on them" (p. 732). The chapters of this book have presented rich descriptions and analyses of actual ideas that students built in the realm of combinatorial reasoning. This work has implications for teaching both in the design and

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[1] According to Gray (2000), Hermann Minkowski (1864-1909), whose metric concept (order-*p* geometry) provided the theoretical foundation for non-Euclidean, taxicab geometry, was a close friend of Hilbert and urged him to accept the invitation to speak at the Congress: "Most alluring would be the attempt to look into the future, in other words, a characterisation of the problems to which the mathematicians should turn in the future. With this, you might conceivably have people talking about your speech even decades from now" (as quoted in Gray, 2000, p. 1).

[2] See Grattan-Guinness (2000) for a critical appraisal of "the range of Hilbert's problems against the panoply then evident in mathematics."

sequence of effective tasks and in demonstrating how teachers could productively interact with student ideas.

The global picture depicted in the chapters of this book underscores the need for time to think deeply and discursively. A special issue of *Educational Studies in Mathematics* collected several analyses concerning discourse in mathematics classrooms. Commenting on these studies, Seeger (2002) wonders about the possibility of a grand, panoramic theory of learning. In arguing for a comprehensive theory of mathematics education, he suggests that such a theory needs to embrace four metaphors of learning that form the axes "social—individual" and "construction—acquisition," and represents them in a two-by-two grid (p. 289).[3] In addition, Seeger further suggests that "theoretical work has to be balanced by the systematic development of focal problems for practice, theory, and research in mathematics education" (p. 289). He proposes two focal problems for mathematics education, one concerning ecological validity and representation and the other referring to the question of time and change.

Beside epistemological concerns, the question of time and change also concerns methodological issues. Building ideas and understanding are certainly temporal and unbounded. Consequently, there are complex judgments an investigator has to make when inquiring into what learners build, understand, or acquire from a discussion or lesson on a particular issue. When does an investigator examine of what learners say, do, and write? Should these actions be examined in the immediate proximity of the discussion or lesson, in some other, more distinct time, or in some combination of these times? Ball and Lampert (1999) raise somewhat similar questions in a study of teaching practice.

Epistemologically and methodologically, this book contributes to understanding the relation between time and development. As outcomes of individual and collective constructive actions over the course of the longitudinal study, the participants build ideas, reason, and employ heuristics to resolve various tasks. They reveal and make salient the important relationship between time and development. The processes by which the participants build their ideas evidence an epistemological reality: knowledge construction is often a slow process. Mathematical ideas do not develop instantaneously and robustly but rather emerge slowly and in their nascent state are rather fragile. Ideas dawn and mature over time. To loose fragility, among other things, ideas need to be reflected on deeply, presented publicly, submitted to challenge, available for negotiation, and subject to modification. That is, the essence of developing and understanding mathematical ideas is often a protracted, iterative, and recursive phenomenon, occurring over more time than is usually appreciated or acknowledged in practice in classrooms and in reports in the literature (Pirie & Kieren, 1994; Seeger, 2002). If learners are to develop deep understandings that are less fragile and more durable than is often witnessed by teachers in schools, they need to be offered extended periods of time to wrestle with a problem as well as to debate and negotiate heuristics, to articulate and justify their results, and to have their ideas challenged and then defend or modify their ideas.

If we agree that students must be actively and purposely engaged in their learning so that they can take ownership and be proud of their accomplishments, we need to create opportunities for this to occur. For example, strands of investigations can be integrated into the regular curriculum as enrichment. When we eliminate the pressure of testing and grades, students can invest in thinking and reasoning for its own sake and for the intrinsic rewards that *knowing deeply* entails. Perhaps every several weeks, within particular strands, investigations can be revisited, and students can bring their more recent, accumulated knowledge to a more sophisticated examination of earlier solutions, and thereby extend their knowing. A focus on justification as a strand of school mathematics has great potential for building a solid foundation for the later study in many fields and certainly of mathematics. A focus on reasoning and sense making is an important requirement for a productive, responsible citizenry. Questioning, challenging, analyzing, revisiting all lead to better ways of knowing. Can we as educators meet the challenge of educating thoughtful students who are motivated by sense making and the critical review of ideas? The challenge awaits us.

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[3] Here Seeger (2002) differs from Sfard's (1998) theorization in which she argues for two metaphors—acquisition and participation or construction—that conceptualizes perspectives on learning and in which she claims that though complementary they are mutually not amenable to critique.

videos and accompanying objects that provided the data for this book and join our expanding community of researchers by providing additional study and analysis.