Chapter 15: Adults Reasoning Combinatorially

Barbara Glass

Date and Grade:	1998 – 2000; College Freshman
Tasks:	Pizzas and Towers
Participants:	Danielle, Donna, Errol, Jeff C., Linda, Lisa, Mary, Melinda, Mike C., Penny, Rob, Stephanie C., Samantha, Steve, Tim, Tracy, and Wesley. (We use the initial C for "college" for Jeff, Mike, and Stephanie to distinguish them from the elementary students of the same names discussed in other chapters.)
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15.1 Introduction

In the preceding chapters of this book, we have provided considerable evidence showing elementary and secondary school students' success in solving open-ended problems, over time, under conditions that encouraged critical thinking. In this chapter, we address the question as to whether similar results can be achieved by liberal-arts college students within a well-implemented curriculum that includes a strand of connected problems to be solved over the course of the semester. From a perspective of conceptualizing reasoning in terms of solving open-ended problems, it was of interest to learn whether students in a liberal-arts college mathematics course could be successful in providing arguments to support their reasoning and in making connections in a problem-solving-based curriculum.

Students enrolled in college-level mathematics courses might be expected to have already developed effective reasoning skills. Unfortunately this is too often not the case. This may be explained, in part, by a history of mathematics instruction in settings that devalue thinking and focus on rote and procedural learning.

Often, in traditional mathematics classrooms, the answer key or the teacher is the source of authority about the correctness of answers; unfortunately, quick, correct answers are often valued more than the thinking that leads to the answer. Too often, teachers ask students to explain their thinking only when answers are wrong, emphasizing the product rather than the process of problem solving. Sanchez and Sacristan (2003) offer data to support this from studying students' written work. They report that students are not accustomed to expressing mathematical ideas, and they offer as an explanation that the emphasis in schools is mainly on producing correct solutions. One consequence is that students tend to develop the belief that all problems can be solved in a short amount of time. Students often stop trying to build a solution if they are unable to solve a problem immediately. For example, in a survey, high school students were asked to respond to the question "What is a reasonable amount of time to work on a problem before you know it's impossible?" Schoenfeld (1989) reported that the largest response was twenty minutes and the average time was twelve minutes. Further, students view school mathematics as a process of mastering formal procedures. These rules are often removed from real life experience and application. As a result, students can feel that answers need not make sense. It is not surprising, then, that students accept and memorize what they are told without making any attempt to deal with meaning (Schoenfeld, 1987).

Since many of the students in this study were previously taught mathematics in this fashion, it would not be entirely surprising if they were unable to apply knowledge from previous mathematics courses to novel situations. Moreover, since a student's willingness to think about a problem is influenced by notions about what mathematics is and what should be expected of students, it is not surprising when students do not display the level of reasoning of which they are capable.

In this chapter, we examine how a small group of community college students enrolled in a liberal arts mathematics class solved open-ended non-routine problems in which they had to build and justify a solution. The tasks were the towers and pizza problems and extensions of these tasks. Our questions were: (1) How do college students solve non-routine mathematical investigations? (2) How do college students' representations and level of reasoning contrast with those of younger students from a longitudinal study engaged in the same investigations? (3) What connections, if any, do the college students make to analogous problems and to the rules leaned in previous classes? (4) To what extent, if any, do the college students justify and generalize their results?

15.2 The Study

The study was conducted in a mathematics class for liberal arts majors called Mathematical Concepts. The curriculum includes algebra and problem solving. Liberal arts students also took a second mathematics course called Contemporary Mathematics that introduces logic, counting methods including combinations and permutations, probability and statistics, geometry, and a cluster of applications called "consumer math." The two liberal arts mathematics courses can be taken in either order, so some students in Mathematical Concepts had already taken Contemporary Mathematics and others had not. Most of the students in the Mathematical Concepts class take the course to fulfill the mathematics requirement, although a few take the course as an elective.

The mathematical background of the students in this study varied widely. Some had taken college preparatory mathematics in high school, while others took only general mathematics courses. Some had already taken other college level mathematics courses, while for others this was their first college level mathematics course. When asked on a questionnaire about their mathematics background, many students described themselves as being very poor mathematics students who disliked and feared mathematics, while others stated that they liked mathematics and had always done well in mathematics classes. There was also a wide range of ages, some students having recently completed high school, with others not having taken any mathematics for many years.

The study took place in a relatively new community college of moderate size in an area of New Jersey that ranges from rural to suburban with very little racial or cultural diversity. In the fall semester of 2000 there were 929 full-time and 1357 part-time students enrolled. As with other community colleges, some of the students attend because poor academic records prevent them from being accepted elsewhere. Others are excellent students who attend the college for a variety of reasons including lower costs and the convenience of being close to their homes and places of employment.

Nine classes ranging in size from six to 25 students were studied between 1998 and 2000. Sections of the course met for fifteen weeks for two 75-minute classes each week or three 50-minute classes each week. The students spent approximately half of the class time working on various non-routine problems in a small group setting. After they worked together on these problems the students were encouraged to present their solutions to the class. In addition, a weekly problem-solving homework assignment was given. As a part of the assignment, students were required to give a written explanation of their solution method and a justification of how they knew that their solution was correct. Students also submitted write-ups of the problems done in class.

Two groups from each class were videotaped as they worked on the towers and pizza problems. In addition, task-based interviews with ten representative students were videotaped**.** Students were selected because they were willing to be videotaped while participating in problem-solving sessions and willing to participate in videotaped follow-up interviews.

15.3 Student Solutions

The students worked on the towers problem during the eighth or ninth week of the semester. By this time, they had become accustomed to working on problems and to justifying their solutions. The students began by working on the four-tall towers problem. They then were asked to consider the five-tall towers problem. Some groups also worked with three-color towers problems.

The students worked on the pizza problem during the thirteenth or fourteenth week of each semester, first on the four-topping pizza problem and then on the five-topping pizza problem. After they solved the basic problems, some groups were asked to consider the pizza with halves problem, in which a topping could be placed on either a whole pizza or a half pizza.

15.3.1 Towers Problems

Most of the college students used patterns or some other form of local organization immediately, and some immediately imposed a global organization scheme. An organization by cases according to the number of cubes of one color was the method chosen by six students. One group, which started the problem by randomly generating towers using a build and check method, switched to this organization by cases at the suggestion of Jeff C. He said,

> Here, put the ones that have three yellows and a red all together. [Danielle rearranges the towers.] Okay. So now we do three yellows and a red at the bottom, 'cause you don't have that. [Jeff C. builds YYYR and hands it to Danielle.] And the ones that have two and two, put those together. [Danielle rearranges the towers.] Now the ones that have three reds and the other.

The cases were: no cubes of the selected color, then one, two, three, and four of the selected color. All six students who selected organization by cases determined that there were two solid-color towers (one all of one color and one all the other color). All six used a staircase pattern to show that they had found all towers with three cubes of one color and one cube of the other color; refer to Figure 15.x for an example staircase pattern.

The case with two cubes of one color and two cubes of the other color was more problematic. The students used a variety of methods to demonstrate that they had found all towers in this group. Two groups, Melinda's group and Donna's group, stated that they had found all towers because they were unable to find any more. But as the students in these groups spoke to the instructor, they began to organize their towers and move toward a proof by cases. However, both groups still stated that their justification for the claim that they had all towers with two of each color was that they could not find any more.

Three of the students, Lisa, Errol, and Wesley, tried to argue that the number of towers is sixteen because four times four is sixteen. The instructor responded that they needed a reason why the answer should be four times four. Lisa then produced a proof by cases, although she had difficulty justifying the case of two cubes of each color. During her interview seven weeks later, Lisa found an organization that accounted for all of the towers with two cubes of each color.

Wesley rearranged his towers, but he offered no explanation for why his arrangement produced all possible towers. About seven weeks later, during the interview, Wesley produced a similar arrangement and used it to account for all possible combinations with a proof by cases where his cases were: 1) towers with four cubes of the same color together, 2) three cubes of the same color together, 3) two cubes of the same color together, and 4) no cubes of the same color together.

Errol's partner, Mary, offered a proof by cases. However, Errol wanted a proof that his numerical argument worked. As he continued to think about the problem, he rearranged the towers in a way that he thought showed that four times four was the correct answer. This arrangement grouped towers with a red on top together and towers with a yellow on top together. When the instructor continued to question him as to why this showed that the answer should be four times four, Errol turned to simpler cases in an attempt to verify his numerical argument. He then noticed the doubling pattern and used that to developed an argument by induction, abandoning the four-times-four argument.

Five of the college students did a proof by cases for the five-tall towers. Each of these proofs referred to opposites (pairs of towers with opposite colors in the same positions). They also all used a staircase pattern to account for the towers with one cube of one color and four cubes of the other color. They used a variety of methods to justify the cases with two cubes of one color and three cubes of the other color.

After Rob and his group had organized their towers by cases, they noticed that the number of five-tall towers was double the number of four-tall towers. They extended the doubling pattern to predict how many towers they would get if the towers were three tall, two tall, and one tall. They then built the one-tall and two-tall towers in order to test their theory. While justifying their answer to the five-tall towers problem, they referred both to their doubling pattern and to a proof by cases (Glass, 2001). The instructor asked the students to think of a reason why the number of towers doubled. After a few minutes, Rob explained to Steve that the number doubled because you could add either a red cube or a yellow cube to the bottom of each tower. He explained as follows, building from a generic original tower he called X.

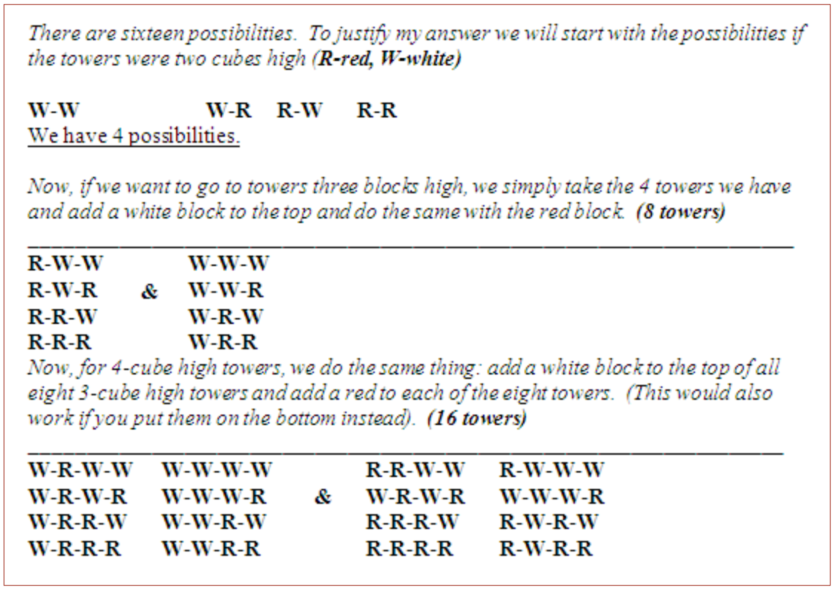
> Okay, let's say the top of our tower is X, X. [Rob writes an X on his paper.] Then we're putting one on the bottom. For every X we can have a Y [yellow] down here, or for every X we can have a red [R] down here. So for each block we have, there are now two more things it could be. So before we just had X. This is X. [Rob picks up the solid red tower of four as an example.] Now we have XR and XY derived from this. XY and XR. [Rob holds up RRRRY and RRRRR.]

Steve demonstrated that he understood Rob's explanation by using Rob's procedure to build two-tall towers by adding cubes to the bottom of one-tall towers.

Wesley built his five-tall towers by adding a red cube to the top of each of his towers of four. He then built the opposites of these towers to find all towers with a yellow cube on top. He justified that he had found all five-tall towers with an inductive argument, but he was unable to extend this reasoning to predict how many six-tall towers there are. However, during an interview seven weeks later, Wesley correctly extended the doubling pattern beyond the case that went from four-tall to five-tall and predicted that there would be 64 six-tall towers.

Jeff C. applied the fundamental counting principle to predict that there would be 32 five-tall towers. After Jeff's group had produced those 32 towers, he used an inductive argument to show that they had found all possible towers by pairing each of the five-tall towers with the corresponding four-tall tower that would generate it.

Errol used the inductive argument that he had developed while working with four-tall towers to predict that there would be 32 five-tall towers. Even though he did not build the five-tall towers in class, he used an inductive method to produce a list of all five-tall towers on his written assignment. Figure 15-1 shows Errol's method.



*There are sixteen possibilities. To justify my answer we will start with the possibilities if the towers were two cubes high (**R-red, W-white**)*

W-W W-R R-W R-R

We have 4 possibilities.

*Now, if we want to go to towers three blocks high, we simply take the 4 towers we have and add a white block to the top and do the same with the red block. **(8 towers)***

R-W-W W-W-W
R-W-R & W-W-R
R-R-W W-R-W
R-R-R W-R-R

*Now, for 4-cube high towers, we do the same thing: add a white block to the top of all eight 3-cube high towers and add a red to each of the eight towers. (This would also work if you put them on the bottom instead). **(16 towers)***

W-R-W-W W-W-W-W R-R-W-W R-W-W-W
W-R-W-R W-W-W-R & W-R-W-R W-W-W-R
W-R-R-W W-W-R-W R-R-R-W R-W-R-W
W-R-R-R W-W-R-R R-R-R-R R-W-R-R

Figure 15-1. Errol's written justification of four-tall towers

Penny, who had been absent the day that the class worked on the towers problem, did the problem at home. She invented a tree diagram strategy to produce an inductive argument for the four-tall and five-tall towers. Her written work is shown below. Her diagram is shown in Figure 15-2.

> *The answer is* $2^4 = 16$ *because my first cube will be either blue or brown (2 choices) my second cube will be either blue or brown matched to a blue or brown first cube (4 choices). For each of those combinations I can use either a blue or a brown cube, doubling my possibilities to 8, and for each of those eight combinations I can add a blue or a brown cube which finally doubles my answer giving me 16 possibilities. (I am using big 'B' for brown and a little 'b' for blue.)*

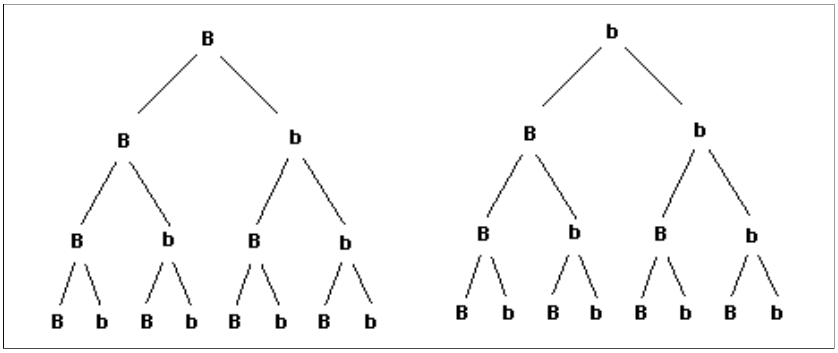


Figure 15-2. Penny's written justification of her answer to the four-tall towers problem

Tim used a binary coding system to justify his conclusion that he had found all possible combinations. This is the same method used by tenth grade Mike from the Rutgers longitudinal study to justify his assertion that there were 32 different five-topping pizzas.

Several groups also had time to work on the three-color towers problem. Mike C.'s group and Rob's group worked on four-tall towers, while Jeff's group worked on three-tall towers. Rob and his partners applied the inductive method that they had developed for towers with two colors to solve the problem quickly. Jeff C. used the fundamental counting principle to calculate the number of towers, but he did not use an inductive method to build the towers. Mike's group divided the problem into two cases: (1) towers with at most two colors, and (2) towers with all three colors. Each student in the group used two of the three colors to build all 16 possible four-tall towers that contained those two colors. After eliminating the three duplicate one-color towers that they had produced using this method, they had a total of 45 towers. Then they worked on the second case: towers that contained at least one cube of each color. They each chose one of the colors and built all 12 towers with two cubes of that color and one cube of each of the other colors. Their solution for the three-color four-tall towers problem was thus 81.

15.3.2 Pizza Problems

All the college students used a justification by cases approach to the pizza problem. The students created their two-topping lists systematically; they held one topping fixed and paired it with each of the other toppings. Then they moved to the next topping on the list. Some students paired each additional topping only with toppings that were below it on the list, reasoning that they had already accounted for the other combinations. Other students considered all pairs and then eliminated the ones that they already had. For the three-topping pizzas, some of the students again systematically went down the list of toppings, while others failed to exhaust both major and minor items. For example, after some students combined pepperoni and green peppers with all other possibilities, they moved to the pepper and mushroom toppings instead of exhausting all possibilities that contained pepperoni.

Two students used a chart that was similar to the chart that fourth-grader Brandon had created when he did the pizza problem (Maher & Martino, 1998). Instead of the ones and zeroes that Brandon had used, these students made a check to indicate that a topping was on a pizza and left a blank space to indicate toppings that were not on the pizza. Interestingly, this is the same method that Brandon's partner Colin used.

Stephanie C., who was simultaneously enrolled in a statistics class, hypothesized that she could calculate the number of pizzas using combinations formulas. She and her partner Tracy systematically created a list of pizzas and compared the numbers to those that Stephanie C. had conjectured using her formulas, confirming Stephanie's prediction.

Several other students who had previously studied combinations tried to calculate the number of pizzas using combinations formulas. However, they did not understand combinations well enough to apply them to the problem correctly. For example, Melinda stated that the problem could be solved either by combinations or by permutations, but she could not remember which to use. Also, students who tried to use formulas attempted to do a single calculation instead of doing separate calculations for each number of toppings. In short, most students were not successful at using combinations. The correct combinations formula for the number of pizzas having exactly *r* toppings when there are *n* toppings to select from is:

To find the total number of four-topping pizzas, therefore, it is necessary to sum ${}_4C_0$ through ${}_4C_4$ (the number of pizzas with exactly 0 through 4 toppings).

15.3.3 Connections Between Problems

During the pizza problem session, several students noticed a relationship between the towers problem and the pizza problem. For example, Rob and Samantha explained as follows:

ROB: So we decided the toppings are the block positions.
INSTRUCTOR: Okay. So you have pepperoni at the top.
ROB: Right. Because it was convenient. So onion would be the second block, sausage would be the third block, and mushroom would be the bottom block.
INSTRUCTOR: Okay. What would your colors be?
SAMANTHA: Orange and yellow.
ROB: Yeah.
INSTRUCTOR: What would orange be?
ROB: Orange means it's on the pizza.
INSTRUCTOR: And yellow?
ROB: It's off.
SAMANTHA: Off the pizza.

Jeff C. was not able to explain the isomorphism during class, but he did explain it a week later during an interview.

JEFF C.: What you could do to relate that to the topping problem. Is that, you could say, you could designate each spot for a topping. This is the pepperoni. This is the green pepper spot, and this is the sausage spot. [Jeff C. writes toppings by the drawing of the first tower.]
INSTRUCTOR: Okay.
JEFF C.: Or onion, or whatever you want to have it.
INSTRUCTOR: Whatever.
JEFF C.: So with nothing on it, with no toppings, there's only one possibility. [Jeff C. points to the first tower he drew.] Now for one topping, there's

three possibilities for a one topping pizza. There's pepperoni, [Jeff C. marks the first block of the second tower.] There's green pepper [Jeff C. marks the second block of the third tower.], and there's sausage. [Jeff C. marks the bottom block of the fourth tower.]

INSTRUCTOR: Okay.

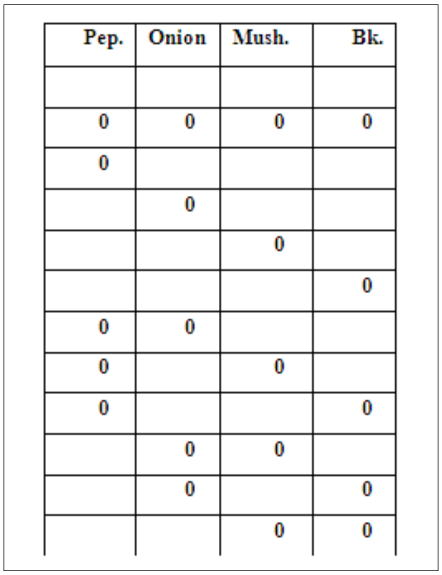
JEFF C.: Then for a two topping pizza [Jeff C. draws three more towers.], there're three possibilities. [Jeff C. moves the actual towers with one white cube.] Pepperoni and green pepper [Jeff C. marks the first and second blocks on the first tower.], pepperoni and sausage [Jeff C. marks the first and third blocks.], and green pepper and sausage. [Jeff C. marks the second and third blocks.] Okay. All right. Which is where the blue, the blue blocks are. [Jeff C. points to the towers with one white cube.] So in other words, if you took, you couldn't flip that one around [Jeff C. takes WBB and turns it upside down and puts it next to BBW.] because then you'd have two of the same combination. You'd have two pizzas with pepperoni and green pepper.

INSTRUCTOR: Okay.

JEFF C.: So those are the three possibilities. The ones with two toppings. And then for three toppings, there's only one possibility – pepperoni, green pepper, and sausage [Jeff C. draws another tower on paper and marks all the blocks.], or blue, blue, and blue. [Jeff C. indicates the solid blue tower.]

Rob's group tried to relate the pizza toppings to colors in the towers problem; they conjectured that each topping corresponded to a cube of a specific color. This is the same explanation that the third-grader Meredith had originally given for the relationship between the two problems. Meredith later revised her explanation to note correctly that the blocks in the towers corresponded to the toppings and the colors in the towers corresponded to the presence or absence of each topping (Maher & Martino, 1999). Unlike Meredith, however, the students in Rob's group did not take their exploration any further and so did not revise their explanation.

Melinda, Stephanie C., Wesley, and Lisa mentioned the relationship between the two problems during their interviews, approximately one week after the class session on the pizza problem. At this time, each student displayed an understanding of the isomorphism between the two problems. During her interview, Lisa first built the four-tall towers and then she produced a pizza problem chart similar to the one that she had made in class. As she completed the chart, she discovered that the rows of the chart looked like the towers. As a consequence, she was able to match the towers with the rows in her chart. A portion of Lisa's chart is shown in Figure 15-3.



Pep.	Onion	Mush.	Bk.
0	0	0	0
0			
	0		
		0	
			0
0	0		
0		0	
0			0
	0	0	
	0		0
		0	0

Figure 15-3. Lisa's chart for the pizza problem

Lisa remarked that the entries in her chart were like the binary system that the class had studied earlier in the semester.

15.3.4 Connections with Pascal's Triangle

While working on the pizza problem, Mike's group noticed the doubling pattern and verified that it continued to hold for smaller numbers of available toppings. As Mike C. thought about why the number of pizzas should double, he discovered that the numbers from the pizza problem matched the rows of Pascal's triangle. He was unable to explain, however, why the addition rule of Pascal's triangle applied to pizzas.

After Rob and Steve's group completed the four-topping pizza problem, Steve remarked that they should look for a pattern before moving on to the five-topping problem. Rob noticed that the pizzas from the four-topping problem formed a row of Pascal's triangle. After figuring out how the addition rule of Pascal's triangle worked with pizzas, Rob used an inductive method and Pascal's triangle to create a list of pizzas for the five-topping pizza problem. Rob's explanation of the addition rule for the Pizza Problem was similar to that used by Mike and the other students in the Night Session, discussed in Chapter 12. Rob wrote:

> *The reason this [Pascal's triangle] works is because every time we add another topping we are increasing the possibility of choice, without losing the old ones. In other words, all the two topping pies in the two topping total still apply when there are three total toppings. Also all those that had two toppings, by adding a third topping, are now three topping pizzas. In this way we can see absolutely, positively, without a doubt, that we have all possible combinations – each new row is built by adding the old columns with the new topping. This once again is the Pascal's triangle principle of adding old combinations with new possibilities to find new combinations.*

Rob's drawing of Pascal's triangle is shown in Figure 15-4.

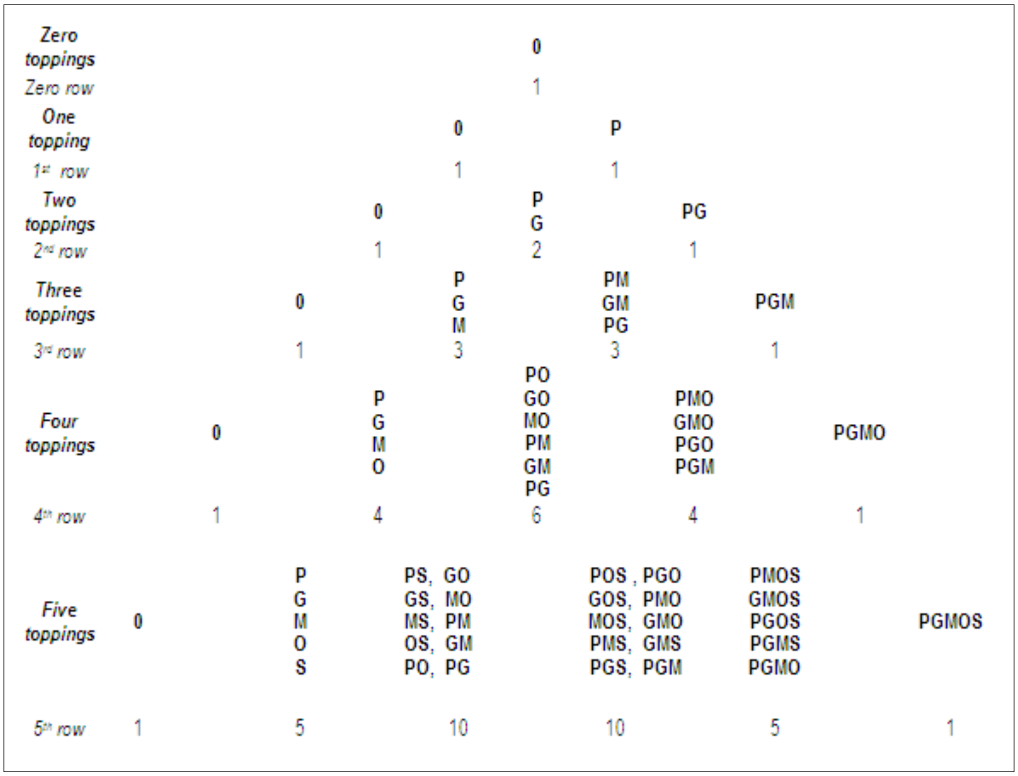


Figure 15-4. Rob's drawing of Pascal's triangle

15.4 Discussion

The college students solved problems and justified their solutions in many of the same ways as the Kenilworth students had done in elementary school and high school. However, the college students solved the problems more quickly than the third and fourth grade children and, unlike the elementary school children, the college students did not generally rely on random checking as a primary strategy. All college students were able to solve the four-tall towers problem and start the five-tall towers problem within a fifty-minute or a seventy-five-minute period. Most also finished the five-tall towers problem within the same period. Several students also had time to work on extensions of the problem. However, the college students also showed less inclination than the Kenilworth students to think about problems for extended periods of time. Many stopped thinking about the problem after they had arrived at an answer, even when the instructor asked them to spend some more time considering the problem in order to reveal their thinking.

After Rob and his group had built their towers and organized them by cases, they noticed the doubling pattern and they developed a proof by induction. This is similar to what Milin had done in the fourth grade. (Refer to Chapter 5 for details.) Mike C. also noticed that the number of towers and pizzas was doubling, but Mike C. was unable to provide a reason for the doubling pattern. It is interesting to note that in fourth grade both Stephanie and Milin had noticed the doubling pattern in the towers problem. Stephanie's progression from pattern recognition to development of an inductive argument took about eight months while Milin's understanding developed more quickly (Alston & Maher, 1993; Maher & Martino, 2000). Perhaps Mike C. would also have recognized the reason for the doubling pattern if, like Stephanie, he had been given an extended time frame in which to develop his ideas. Mike C., however, was limited to about eight weeks to develop his ideas about the problem.

The college students made very few connections with previous mathematical knowledge, and most of these connections were trivial connections. Several students recognized that the problems were related to permutations or combinations, but most did not understand these concepts well enough to correctly apply them to the solutions of novel problems. This would suggest that learning about mathematics in an atmosphere in which students are told what to do does not enable them to develop genuine understanding. In contrast, the students in this study did demonstrate a high level of reasoning as they thought about the problems, justified their solutions, and made connections between problems and the mathematics they learned within the course.

Some of the conditions of the Rutgers University longitudinal study, such as extended classroom sessions and revisiting the same problem several times within an extended time frame, could not be replicated because of the time constraints of a college course. However, many of the conditions that enabled the elementary and secondary students to become thoughtful problem solvers were duplicated. Both groups were given rich mathematical tasks and were encouraged to explain their reasoning and methods of solution and to justify their solutions to the problems. Both groups were engaged in thoughtful mathematics. They found patterns, developed methods of justifications, and provided justifications that their patterns were reasonable. It cannot be disputed that the students in the Rutgers longitudinal study benefited from exposure to rich mathematical experiences over an extended period of time. It is also significant that the students in this study, who had previously experienced a variety of generally traditional mathematics instruction, demonstrated that it is not too late to introduce rich mathematical experiences in a collegiate level mathematics class. The level of reasoning that these students demonstrated provides evidence that it is possible to experience thoughtful mathematics within a traditional fifteen-week college semester.

In this chapter, we have shown how college students worked on the towers and pizza problems. In the following chapter, we will follow up on the work on these college students and other college students, comparing the work of these college undergraduates to the work of the longitudinal study high-school students (see Chapter 8) on the extension of the towers problem called Ankur’s Challenge (finding the number of four-tall towers, built when choosing from three cube colors, having at least one cube of each color).