Chapter 14: "Doing Mathematics" from the Learners' Perspectives

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Date and Grade:	1999 – 2000; high school
Tasks:	Clinical interviews
Participants:	Ankur, Brian, Jeff, Mike, and Romina
Researcher/Instructor:	John Francisco

14.1 Introduction

The previous chapters focused on aspects of the cognitive development of the students in the longitudinal study. The present chapter looks into the epistemological growth of the students. During the longitudinal study, individual clinical interviews were conducted with the students with the goal of capturing the mathematical beliefs that the students might have developed in connection with their experiences in the longitudinal study. This chapter reports on the analysis of five such interviews. The results provide insights into the students' views about mathematics and about how it should be learned and taught. The findings challenge the widespread view that students below college hold naïve epistemological views; support studies that show that students who experience constructivist leaning environments tend to develop sophisticated epistemological beliefs, and highlight the important of past mathematical experiences in framing individuals' mathematical beliefs.

Research on students' views about mathematics can be placed within the field of *personal epistemological beliefs*. This is a field traditionally concerned with describing individuals' views about the nature of knowledge and knowing. A substantial amount of research has been conducted within the field since the pioneering work of Perry (1970) with Harvard college students. Even more research has been associated with this field since the epistemological construct was expanded to include individuals' views about learning, teaching, and intelligence through the work of Schommer (2002; 1995) and some continental scholars (De Corte, Eynde, & Verschaffel, 2002; Pheknonen, 2002). Even though the expansion has not been free of controversy, it is recognized that this expansion has brought the research on the field closer to classrooms practice.

Students' epistemological beliefs have been examined in relation to a variety of constructs. Students' beliefs have been studied in relation to the students' home and school environments (Hammer & Elby, 2002) and their teachers' epistemological beliefs (Hofer, 1994; Lyons, 1990; Pirie & Kieren, 1992; Roth, 1994). There has been also extensive research that has examined students' beliefs within disciplines (Carey and Smith, 1993; Ceci, 1989 Lampert, 1990; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993) as well as across disciplines (Case, 1992; Sternberg, 1989). However, a comprehensive review of the field (Pintrich & Hofer 1997; 2002) suggested that a number of challenges remain to be addressed. One particular challenge is the need for more research on the epistemological beliefs of students below college level. Except for a few cases (Schoenfeld; 1989; Pehkonnen, 2002), most research in the field has remained at the college level. The review notes that there have been few studies involving students below college and even fewer below high school. The review further points out that lack of such research has resulted in students below college being assigned naïve epistemological beliefs only because research findings show that entering college students tend to hold such views. Another challenge is the lack of studies that have examined the epistemological beliefs of students who have experienced a constructivist learning environments. The few existing studies (e.g. Hofer, 1994) have been exploratory in nature.

The present study grew out of a natural interest on the part of the researchers to examine the epistemological growth that participating students in the longitudinal study might have experienced in connection with the particular conditions in which they were asked to do mathematics. The researchers were particularly interested in the students' beliefs about (1) success and failure in mathematics, (2) knowing mathematics, (3) learning and teaching mathematics, and (4) how the practices that they assigned to doing or learning mathematics compared with those in other disciplines. However, the researchers also sought to make a contribution towards deepening the research community's understanding of the epistemological beliefs of students below college, particularly at the high school level, and of students who experienced constructivist mathematical environments. The researchers viewed the longitudinal study as a "learning experiment," rather than a "teaching experiment," through which they tried to understand how students construct mathematical ideas while working on opened-ended mathematical tasks in particular conditions. However, there were no preconceived ideas about what students were to learn or how they were supposed to learn. Students' constructed mathematical ideas and reasoning were results, not preconceived goals, of the research. This was consistent with a constructivist approach to learning in the sense that participants had plenty of opportunities to construct and accordingly revise their ideas without any guidance from the researchers, but rather within their own community of learners.

The students were interviewed about their experiences in the longitudinal study, and from these interviews, inferences are made about their mathematics beliefs. Their conversations provide insights on their mathematical beliefs and challenge the widespread view that students below college level hold naïve views in contrast to studies that show students who experience constructivist leaning environments tend to develop sophisticated epistemological beliefs. However, the results also highlight the importance of past mathematical experiences in the development of individuals' mathematical beliefs.

This study used a phenomenological approach to the students' experiences in the longitudinal study. Researchers avoided imposing any interpretive framework on the students (Wilson, 1977) and sought to infer the students' epistemological views among the meanings that the students assigned to their experiences in the longitudinal study (Creswell, 1998). Overall, the approach was similar to Perry's (1970) idea of inferring individuals' epistemological beliefs from their reflections on educational experiences.

Data for the present study consisted of one-hour videotaped individual interviews with the five participating students about their experiences in the longitudinal study. The four males and one female – Ankur, Brian, Jeff, Mike, and Romina – agreed to be interviewed and videotaped. However, it was the students' long experience in the longitudinal study, starting in first grade, that constituted the major criteria for selecting the students to take part in of the interviews. Their long-term participation in the longitudinal study satisfies the *criterion sampling* method (Miles & Hubberman, 1994) recommended for phenomenological studies.

The interviews used a semi-structured interview protocol. There was a clear goal (i.e. capturing the students' views about mathematics as a discipline with particular practices and criteria of validity), but the interview proceeded by eliciting and building on students' reflections on their experiences in the longitudinal study to ascertain the students' epistemological views. Typically, the interviews started with the open question, "What are your memories of the longitudinal study?" Then the researchers tried to steer the interviews towards obtaining insights on the students' views on mathematics.

14.2 Findings

The search for answers to the research questions generated five major themes about personal success and failure in mathematics, mathematics as sense making, mathematics as a discovery activity, mathematics as an activity involving discourse, and the relationship between mathematics and other disciplines. These themes are described below, along with supporting statements from the students. (Emphasis was added to quotes.)

14.2.1 Personal Success/Failure in Mathematics

All of the students described themselves as confident and good in mathematics. Ankur even said, “I’m well above average in mathematics,” and modesty prevented Mike from describing himself as being better than a “normal kid.” There were differences, however, on how the students construed mathematical success. Mike and Ankur emphasized personal interest and hard work as the ultimate sources of success. They argued that those who like mathematics can be successful because they are willing to work harder in mathematics than those who do not like it. The other students stressed the importance of previous mathematical experiences and training. In particular, they singled out particular aspects of experiences in the longitudinal study such as collaborative work and opportunities to come up with ideas, as opposed to merely receiving them from teachers or experts, as having contributed to their success and confidence in mathematics. Romina further suggested that confidence and success is built over time:

> In fourth grade, I didn't know who you were. Now we're comfortable with you. You've been our teachers for ten years. That's what you've been to us, so now it's easier, and we know what's expected of us, what we have to do. Before we would wait for you to give us a little start or a little push and point us in the direction. *Now you hand us a problem and you just kind of leave, and we just do it ourselves*. We just start experimenting and see what we can give you.

She also suggested that lack of success can be a function of how success is defined. She explained that, although she generally felt confident in her abilities, she might not feel confident in situations where she is asked to engage in rule-based mathematics, as in textbooks, as opposed to ways that are personally meaningful:

> They might throw out, “Oh, do you know this rule?” I’m like, “No, but if you sit me down, maybe I know it.” I know it in my own way, not in their way. Everything I explain is in my own words, not in anyone else’s words. It’s not from some mathematician from a thousand years ago, because I don’t know that. I didn’t know what the pyramid [Pascal’s Triangle] was called. I just know everything in my own way. Everything has Romina’s definition to it.

There were no suggestions that the students considered success as a quality or trait that people are born with. On the contrary, a closer analysis of the students’ reflections suggests that the students converge on recognizing the importance of past mathematical experiences in promoting mathematical confidence and ability either directly or indirectly through motivation. Romina’s last statement also suggest that standardized testing has the potential to portray otherwise bright students as mathematically weak only because the students do not do mathematics in the ways prescribed in textbooks or by experts.

14.2.2 Knowing Mathematics as Sense Making

The students’ reflections on their experiences in the longitudinal study emphasized the importance of understanding as opposed to memorization of concepts. For example, Mike reported gaining increased conceptual understanding in the longitudinal study:

> It feels different now because I know a lot more than I did before. If I were to solve the same problems, it would be easier. *I understand a lot better too the whole concept behind each problem*. Like, all the problems that have been given to us, I feel like, somehow, one is related to each other. When you’re little, you can’t really understand that.

Romina emphasized the importance of building durable understanding. In particular, she explained that it involved the ability to recall as well as reconstruct previously learned mathematical concepts:

> Because everything I do I understand, because it's more than just the numbers to me. *If you understand something from the beginning, you're going to always understand it*. You can't forget something like that. And like an equation, I don't really know any equations. It's like things, *I don't know any solid equations, but I could explain to you something and work from there*. And you're likely to forget an equation.

Jeff related understanding to the ability to “explain” what one knows to others:

> *The name really doesn't matter*. That's neither here nor there. I mean, just knowing how to do it, that's the important part, that's what we learned. And that's being able to do it, being able to teach it to somebody else, *to explain it*, to use it for what you need to use it for. That's what really matters, not being able to know the name of it, or how to draw it up, or anything like that.

Brian emphasized the importance of developing the ability to “look deeper than just the surface” and of always asking “why,” qualities which he asserts that he gained in the longitudinal study:

> When I look back at things, I'm happy I got involved in this program. Because, I know at times, I seem very frustrated with it. But if I think hard, I really have gained a lot of knowledge, and *I learned how to look into things deeper than just surface things like, "Why is it like this?" Now, I start thinking like that.* And it helps me compute things in my mind better. Like, I really don't know how to put this, but it just helps me in doing things other than math. I think more "in-depth" and very seriously about things.

Ankur's idea of understanding was not as explicit as that of others. However, when recalling different mathematics experiences in different schools, he was clear about why he liked the one in which the teacher did not "teach out of the textbook": it promoted understanding.

> At Harding [Elementary School], my math experience was, I'd say it was good. *The teacher would teach, I'd understand*, I'd participate, and it was just, I enjoyed it. And then we went over to the Springfield [Regional High School], and I did not enjoy geometry class at all. It was one of the first times that we used the textbook. I don't remember in Harding using a math textbook. And the teacher would just simply teach out of the book, and assign homework, straight problems, and it wasn't anything that I enjoyed. Then after Springfield, we came here [to the local Kenilworth High School] and I had Mr. Pantozzi [his mathematics teacher who was also involved with the longitudinal study as a researcher and graduate student] for two years. And I enjoyed that. His teaching style was like none other, and it works.

Figure 14-1. Ankur

In particular, if Romina's statement above further suggests that knowing or understanding mathematics has a personal dimension, Jeff's statement suggests that understanding has a social or interpersonal dimension to knowing mathematics.

14.2.3 Mathematics as a Discovery Activity

Ankur's statement in the previous section suggests that he favored a mathematical environment where students [not teachers] came up with their mathematical ideas or knowledge as opposed to merely receiving them from teachers or textbooks. The implicit idea of learning as a discovery activity was present in the reflections of all students, albeit articulated in slight different ways. Mike argued that discovery learning helped the majority of students understand mathematics and emphasized exploration of concepts over time and group work during the discovery process:

> *Kids can learn new things if they discover them themselves*, and not if somebody tells them, I think that is a better way of learning. Like Mr. Pantozzi, he gives us some information, but basically, he lets us discover the things that a normal teacher would just tell us. Like we were learning about *e* [base of natural logarithms], and he told me that when he was in school, the teacher told them, "*e* is this, 2.7, whatever." The teacher told him what it is. In our class, *all we did was just explore e. We took days at a time*, and *I have a good understanding of it*. I guess, in a normal class, only selected kids might understand it. But in a class where everybody's working together, everybody's a part of the teaching, everybody or at least *the majority of kids will understand it*.

Jeff also made reference to working on tasks over time and to group work, but he stressed the importance of mathematical arguments or discussions during group work. He asserted that participating in discussions was a better way than listening to teachers for students to build durable mathematical understanding:

> Well when the teacher just comes out and tells you the answer, you find you can study it, you can get it for that test, but a few weeks later, a few days later, it doesn't matter anymore, you don't need to know it, and you're onto the next thing. And that's wasting your time. Because you spend the whole year running through this, you learn, say, twenty different things, but by the end of the year, you've forgotten them all, and you have nothing. If you would, say, argue for a couple days or weeks or whatever on different topics, you cover ten things, but when you walk away, you still know those ten things at the end. And that's why it's important to do that, and not just get the answers.

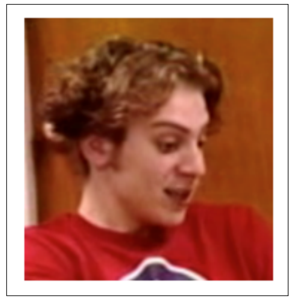

Figure 14-2. Jeff

Romina also singled out mathematical discussions during mathematical activities. In particular, she added that disagreements during the discussions were an important cognitive mechanism by which students built new knowledge and how she, personally, learned mathematics:

> Because if you're, like, passive, and I'm like, "This is what I think it is," and everyone is, "Okay, that's what it is," we all sit back and we all take that and we never go any further. But if I disagree with someone, they'll have to explain it to me, and if they're explaining it, they're either going to find something right, or they're going to find something more. So, if I don't agree with it, they're going to explain it to me, but if they find something wrong, maybe I can help, and then someone else may disagree with me. And that's how we get through everything. We just disagree. I've always had to argue to get somewhere, because they never actually told me where we were heading with anything. *So, through arguing, that's the only reason I know math.*

Brian put the emphasis on hands-on experiences during mathematical activities. He argued that hands-on activities motivated students to do mathematics and helped them build durable understanding of mathematics:

> If I could changes courses, *I would make everyone hands-on* because kids get tired of sitting there. But when you're up doing things, time flies, and you have fun and you learn, which you retain more.

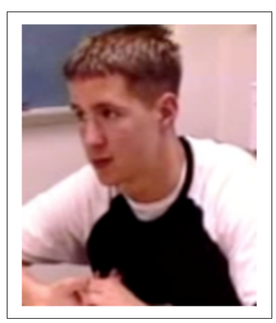

Figure 14-3. Brian

Ankur favored problem-solving activities involving interesting and challenging tasks and collaborative work, as opposed to teaching by the textbook:

> Right now, in my current math class, the teacher doesn't use a book. I'd say *he comes up with problems*, and most of the problems are *interesting problems*, and a lot of them are *challenging*, and all the students participate. *We enjoy working in groups*. We help one another, and that helps out a great deal.

14.2.4 Mathematics as an Activity Involving Discourse

Some of the students' statement in the previous section, particularly those by Jeff and Romina about mathematical arguments, suggest the idea of doing mathematics as a discursive activity. The statements assign mathematical arguing the cognitive role of fostering knowledge acquisition. Jeff's statement about understanding as explaining ideas also suggests the idea of arguments as a way of proving or establishing the validity of mathematical claims. In the statement below, Jeff is even more explicit about the arguments as way of proving mathematical claims:

> *We didn't know if we were right or wrong*. You only knew so much, but I would have my idea about how to get to a certain point and you might have the same idea about how to get to it. But getting there was the hardest part. That is what we were arguing about, the right way to get there, the right way to make sure we covered the basis, how to make sure, *how to prove* what we needed to accomplish.

Mike suggests a similar idea, but puts it in the context of probability. However, he does not claim that arguing as proving only take place in probability. Rather, he suggests that because uncertainty is more common in probability than in any other area of mathematics, arguments are more likely to take place in probability:

> The reason we argued about math, because math is like, when we do about probability, probability is an iffy subject. Like, sometimes, *I mean the math says it's right, but do you believe it's right, and sometimes that influences your decision. That's probably why we argue*. I remember the problem with the World Series Problem [see Appendix A]. We had two different answers. I still don't know which one is correct.

The students had a response for those who might claim that in group work, some students might not be engaged and so might not learn. Above, Mike suggested that discovery learning with group work can help the majority of students learn mathematics. Ankur's statement below suggests a similar idea and tries to illustrate how it happens:

> Usually, you think that only one person in the group is learning, but *if the group fully participates and everyone is involved, everyone in the group learns*. When the Rutgers group comes over here, we all learn. I don't think there is a case when someone doesn't understand. Because if one person doesn't understand, they'll say something, or even if they're quiet, someone else will suggest something, will ask them if they understand, or say "Could you explain it back to me" And that's how everyone learns.

14.2.5 Mathematics and Other Disciplines

The students had different responses on whether the practices that they associated with learning or doing mathematics were specific to mathematics or applicable to other subjects or real life. Jeff claimed that arguing and covering ideas in depth were absent in other subjects such as physics, where teachers just gave out answers and ran though the material:

> *When the teacher just comes out and tells you the answer*, you can get it for that test, but *a few weeks later you don't need to know it*. That's wasting your time because you spend the whole year running through this, you learn, say, twenty different things, but by the end of the year, you've forgotten them all. *If you would argue for a couple days or weeks or whatever on different topics*, you cover ten things, but when you walk away, *you still know those ten things at the end*.

Ankur suggested that he thought that teaching out of the textbook, as opposed to the discovery method, was more suited to other subjects such as history than to mathematics:

> In ninth grade, I was in a different school, and the teacher there taught me differently. She [the ninth grade math teacher] taught more like a history teacher. *A history teacher would simply teach out of the book, just go right down through the years, and you'd learn like that. But the math teacher, I wouldn't think, a math teacher should teach like that*. A math teacher usually teaches differently. I don't know how to explain it, but it just seems that way. This teacher taught straight out of the textbook, you wouldn't learn anything more, just simply what the book stated.

Romina was the most categorical of all the students in her response. She claimed that arguing about ideas was a learning practice specific to mathematics and could not be implemented in other subjects such as English and History:

> *Well, math is where the most arguing is*. Like, you *can't do this in other classes*. It's not like, in English, you read. You don't argue. It's there. It's written. And in history, you don't do the same. In math, it's like, well especially the way I've

been taught, because I have never actually had a math teacher that's said, "This is the equation, put in the numbers, and do it." I've always had to argue to get somewhere.

Figure 14-4. Romina

Brian and Mike, however, had different responses. They suggested that learning mathematical practices were also relevant to other disciplines. For example, Brian suggested that his history teacher also used a problem-solving approach as opposed to just telling students what to do:

> Well, the closest thing to my math class would have to be my history class. My history teacher is an incredible teacher. He always, like for instance, we're doing the Cuban Missile Crisis thing, he set the class up into countries, and *we had to all deal with the problems, instead of just sitting there and telling us*. Next to Mr. Pantozzi, he gets us involved just as much as he does.

Mike argued that his longitudinal study experiences were relevant to other subjects. "I think it's relevant to a lot of other subjects, like science, history; I guess you could apply it to, basically, all subjects" and claimed that he used his experiences in the longitudinal study, which he called a "type of thinking" in real life and other subjects:

> I guess I use the type of thinking in, like other subjects in school; I don't know how you can apply it to life. It's not hard to recognize what style of thinking you're thinking of. I can't compare it with someone else's because I don't know what they're thinking. *So, I think, yeah, I probably do use it in life, and other subjects in school.*

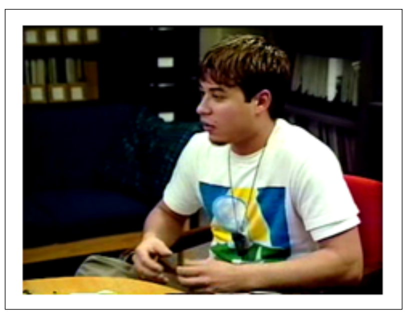

Figure 14-5. Mike

A closer analysis of the statements suggests that differences among the students' responses regarding the applicability of mathematical learning practices to other subjects reflect differences in interpretation of the question asked. Jeff, Romina, and Ankur seem to have answered the question of whether the mathematical learning practices were *actually* taking place in other subjects. Mike and Brian, however, seem to have understood the question as asking whether they believed that those practices were applicable to other subjects.

14.3 Conclusions

The analysis of the interviews with the five students who participated in this study suggests that the students (1) are confident in their mathematical ability, (2) emphasize mathematical understanding over memorization, and view mathematics as (3) a discovery activity and (4) a discursive activity. The results also suggest an agreement that that (5) the practices were not being implemented in the regular schools, except in a few isolated cases (a history teacher and Mr. Pantozzi). The findings suggest a few insights.

The students' emphasis on the importance of learning as a discovery activity suggests that they view themselves as learners as *active participants* in the *construction* and *justification* of their mathematical *knowledge*, and not as mere receivers of knowledge and truth from experts or textbooks written by experts. Within the domain of personal epistemological beliefs, such a view is held by individuals holding sophisticated or powerful personal epistemological beliefs. As a result, the findings of the present study challenge the aforementioned widespread belief that students below college hold naïve epistemological beliefs based on research that show that to be the case among freshmen college students. Given that the conditions of the longitudinal study were consistent with a constructivist approach to learning, the results also support findings from exploratory studies suggesting that students who experience constructivist learning environments tend to develop more sophisticated epistemological views than students who experience teaching approaches based on showing and telling students what to do.

The students' views about mathematics are also consistent with the non-traditional approach to mathematical learning and teaching, advocated by the research community and promoted through publications such as the 2000 *Principles and Standards for School Mathematics* of standards of the National Council of Teachers of mathematics (NCTM). This is evident in the students' emphasis on durable mathematical understanding as opposed to memorization of concepts or procedures, discovery learning, convincing or explanatory arguments, and collaborative work. The students articulate the merits of such practices in enhancing learning for the majority of students and point out that these practices also motivate them to learn. This suggests powerful beliefs within the particular field of personal epistemological beliefs and within the larger field of mathematics education.

Another dimension of the depth of the students' mathematical views is reflected in the nature of the students' articulation of cognitive process involved in doing or learning mathematics. They students provide different characterizations of mathematical understanding concepts with qualifiers such as conceptual, operational, durable, personal, and interpersonal. Understanding is also defined not only as recall but also as the ability to reconstruct previously learned ideas. Arguing is associated with knowledge acquisition as well justification or proof for mathematical claims. There are also rich descriptions of the conditions in which learning, particularly discovery learning, takes place: hands-on activities, explorations, problem solving, interesting and challenging tasks, collaborative work as arguing or discussing ideas, work on tasks over time, and so on. The students' ability to articulate in detail different cognitive aspects involved in learning is another measure of depth of the students' mathematical beliefs. In particular, Romina's idea about knowing or understanding mathematics as personal is particularly insightful. A great deal has been written about the issue under the idea of personal representation (Francisco & Maher 2005; Maher, 2005; diSessa & Sherin, 2000; Davis, 1992; and Davis & Maher, 1990). The idea has been encourage teachers to attend to and promote the conception between formally defined mathematics and students' personal conceptualizations to promote understanding.

Finally, it is particularly interesting that the students' mathematical views mirror the particular conditions within which they engaged in mathematical activities in the longitudinal study. Under the idea that the longitudinal study was more of a "learning experiment" rather than "teaching experiment," researchers encouraged the students to *work collaboratively* with other students; justify their reasoning to classmates; *be the arbiters* of whether or not a solution was correct based on whether it made sense; work on the same tasks over an extended period of time tasks; and revisit similar or same task and refine their ideas and mathematical reasoning. Such conditions are reflected in the students' thoughts about their experiences in the longitudinal study. This suggests that the importance of construing mathematical beliefs within the particular experiences in which students engage in mathematics. In particular, this highlights the importance of teachers paying closer attention to the kind of beliefs that they might be promoting in their students through their conscious or unconscious practices or beliefs in mathematics classroom. This is an area which remains largely unexplored, as few studies have examined the relation between mathematical beliefs and particular settings, whether within or across cultural settings.

In summary, we provide here an existence proof that students below college level are capable of building powerful mathematical beliefs and insights about the cognitive processes and conditions involved in doing mathematics. However, we also emphasize the importance of examining mathematical beliefs within particular learning conditions.

In this and preceding chapters, we followed students in the longitudinal study through elementary school, middle school, and high school, working on problems in combinatorics. In the following chapter, we look at a group of college students working on the towers and pizza problems, and we see how their work compares to that of the younger students.