Chapter 13: "So Let's Prove It!"

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Date and Grade: May 5, 2000, Grade 12
Tasks: The Taxicab Problem
Participants: Brian, Jeff, Mike, and Romina
Researchers: Carolyn A. Maher and Arthur B. Powell

13.1 Introduction

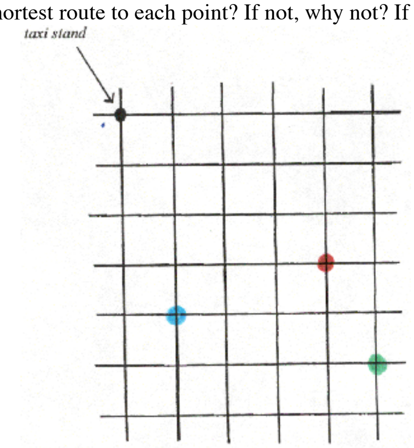
In previous chapters, we observed students throughout middle school and high school working on and making sense of two isomorphic problems in combinatorics – the towers problems and the pizza problems. In this chapter, we see how students just finishing high school work on another isomorphic problem, demonstrating the application of techniques and ways of thinking that they developed throughout their previous years in the study. In this chapter, we further address the challenge that Davis (1992) proposes to mathematics education researchers to investigate the emergence among learners of what lies at the core of mathematics: mathematical ideas. Here, a cohort of four high school seniors — Brian, Jeff, Mike, and Romina — elaborate mathematical ideas and reasoning through work on the Taxicab Problem. They display criteria and techniques for justifying claims and an awareness of the power of generalizing, particularly as an aid to respond to special cases.

13.1.1 The Task

The problem-solving session was held in a classroom during the late afternoon, after school hours. During the session, which lasted about 1 hour and 40 minutes, the four students collaborated on a culminating, performance-assessment task of the research strand on combinatorics — the Taxicab Problem:

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



13.2 Justifying Claims

It is a non-trivial cognitive task for students to recognize which statements or claims in their mathematical discourse require justification or proof. This is particularly true if the students deem the claim to be obvious or if the students are in the midst of group problem solving with intellectual peers. On May 5, 2000, in the late afternoon, after school, and just a few weeks shy of their high-school graduation, Brian, Jeff, Romina, and Mike are seated around three sides of a trapezoidal-shaped table, on top of which are four black felt-tip markers, sheets of blank paper, and a problem statement. The statement is of a problem in which one is to determine in a given rectangular grid the number of different shortest paths between pairs of specified colored, endpoints (black and blue, black and red, and black and green). A researcher asks the four seniors to read the Taxicab Problem and to see whether they understand it. Jeff asks aloud whether one has to stay on the grid lines and whether they represent streets. The researcher responds, "Exactly." Each student has taken a marker. Among themselves, they observe that from the black endpoint or "taxi stand," five and seven are respectively the number of blocks it takes to reach the blue and red endpoints or "pick-up points." Moreover, some assert that different routes to each point have the same length as long as one doesn't go beyond the particular pick-up point. Especially noteworthy from cognitive and pedagogical viewpoints, Brian says to his colleagues, "So, let's prove it!" After a few quiet moments, a discussion ensues as to how they know that their claim is true.

13.2.1 Generalizations, Isomorphisms, and Transitivity

After further individual and collective work and discussions, Brian, Jeff, Mike, and Romina decide that to determine the number of paths between three specialized pairs of endpoints they need to generalize the problem. This moment is a watershed event in their mathematical work on the Taxicab Problem. Through their various heuristic actions, the students generate data that they consider reliable. They reflect on numerical patterns in their array of data, observe that it resembles Pascal's Triangle, and conjecture that Pascal's arithmetic array underlies the mathematical structure of the problem. How do they justify this conjecture? They embark on building an isomorphism between the Towers Problem and the Taxicab Problem since from previous experience they know that Pascal's Triangle underlies the mathematical structure of the Towers Problem. The students' strategy can be interpreted as justifying their conjecture by transitivity: (a) the mathematical structure of Pascal's Triangle is equivalent to that of the Towers Problem and (b) the mathematical structure of the Towers Problem is equivalent to that of the Taxicab Problem; implying that (c) the mathematical structure of Pascal's Triangle is equivalent to that of the Taxicab Problem. The students knew that (a) is true and demonstrate (b) to justify and conclude (c).

13.2.2 Reasoning and Justifying

In the following sections, we discuss students' methods of reasoning and ways of justifying their statements.

13.2.2.1 Realizing the need to discursively build a justification

Two and a half minutes after receiving the task, Romina begins the first student-to-student interaction. It centers on a question about a relation that she notices about which Romina invites her colleagues to comment.

ROMINA:	Isn't it like anyway you go-
BRIAN:	Pretty much, because look-
ROMINA:	As long as you don't go like past it. [Facing Brian's direction.]
BRIAN:	The first one- No.
MIKE:	Well what if you go to the last one-
BRIAN:	You can go all the way down and go over and go down three and go over two. [Tracing the routes above the problem sheet with a black marker in his right hand.]
ROMINA:	Isn't it- Don't they all come out to be the same amount of blocks?
BRIAN:	Five.
JEFF:	Five?
ROMINA:	Five? I got seven.
JEFF:	Uh, which one- Yeah, we were both looking at the red one.
BRIAN:	I'm looking at blue. [Mike is tapping his pen on the grid along intersection points.]
JEFF:	Yeah.
ROMINA:	Oh, okay.
JEFF:	All right. I mean pretty much.
ROMINA:	As long as you don't go like past it you're fine. So it's the same thing.
BRIAN:	So, let's prove it.

Romina's interrogative, "isn't it like any way you go, they [the lengths of routes] all come out the same…as long as you don't go past it [the pick-up point]?" suggests that she is aware of a relation among efficient ("as long as you don't go past it") paths or routes between the taxi stand and the red pick-up point. She observes that as long as one does not go beyond the red pick-up point that the numbers of blocks traversed or lengths of routes to red equal each other. Specialized to the red pick-up point, she expresses three awarenesses about relations among objects: (a) an efficient route will be a shortest route, (b) there can exist more than one shortest route, and, her central observation, (c) efficient routes have the same length. These three ideas are important and fundamental for progressing toward a resolution of the problem task.

At first, Brian disagrees ("The first one- No, 'cause-") and then, examining routes to the blue pick-up point, attempts to understand Romina's remark ("You can go all the way down and go over and go down three and go over two"). Afterward, Jeff and Romina try to understand Brian's assertion, "five," for the number of blocks traversed by shortest routes between the taxi stand and the red pick-up point. Ultimately, Brian sees that they are speaking about routes to the red point ("Yeah, we were both looking at the red one."). While, they understand that he is referring to the blue pick-up point ("I'm looking at blue."). Taking up Romina's observation for the red pick-up point along with his own for the blue point, Brian suggests, "So, let's prove it."

Brian's proposal is not immediately entertained. However, after about 1.5 minutes, Jeff poses a question that places Brian's proposal back onto the agenda, and the students discuss how they know that Romina's unchallenged assertion is true.

JEFF: So why- why is it the same every time?
MIKE: You're going left and right.
ROMINA: Ours is a four by one, right? It's the only way to go.
MIKE: It's the only way you can go. Yeah, it's a four by one, unless you go backwards a couple of times.
ROMINA: You can't go, well-
MIKE: I know that would be dumb.
BRIAN: [inaudible] the shortest route only if you go forward.
MIKE: But the only- You can't go diagonal so you have to go up and down. So if the thing is down this many and
JEFF: Over that many, it's the same
MIKE: It's the same-
ROMINA: It's the same area
MIKE: No matter how you do it, no matter how you do it it's- you have to- you can't get around doing that. [Pointing and gesturing around his grid]
ROMINA: All right.
MIKE: You can't get around going four down and right one 'cause -.
JEFF: All right, yeah. All right.
MIKE: You can't go over there. You can't get around doing that.
JEFF: Yeah.
ROMINA: What if I were to go like to the red when I go one, two, three, four- [Pointing at her problem sheet.]
MIKE: But they're not asking for that.
ROMINA: Five, six, seven.
JEFF: Five, six, seven. It's the same thing.
ROMINA: Like how- how am I going to- like how would I-
JEFF: It's the same thing.
MIKE: It's the same.
ROMINA: -devise an area for that? Like this- this area up here? [Motioning with her pen on her grid, indicating the area of the rectangular space whose vertices are taxi stand and the red pick-up point.]
BRIAN: Like plus and [Inaudible].
JEFF: Well, it's not area.
MIKE: It's not area. It's just a-
JEFF: It's the perimeter. It's like each one being one.
MIKE: One, two, three, four, five, six, seven. [Pointing at Romina's paper and counting the length of a route to the red destination point.] [Jeff scratches his head.]
ROMINA: All right.
MIKE: There's no way you can get around going- [gesturing with his hands]
JEFF: Going seven blocks.
ROMINA: No, yeah, I understand.
MIKE: Across that many and down that many because you can't go diagonally. Can't- [gesturing with his hands over his problem sheet across to the left and then down]
JEFF: Yeah.
MIKE: Can't get around it, so- [gesturing with his hands]
JEFF: I mean, that's the most sensible way I think to say that. Right? And they want to know how many though.

Justifying Romina's observation, reiterated by Jeff, or, equivalently, entertaining Brian's proposal becomes a shared project of the participants. When Jeff poses his question ("why is it the same every time?") and the others understand his "it" to mean "the set of efficient routes to a pick-up point." Mike's immediate response, coming just 4 seconds after Jeff finishes uttering his question, is in contrast to the silence that Brian's proposal received almost two minutes earlier. The ensuing discursive exchange hints that the issue of the why Romina's observation was true in general remained a concern of the participants and that they are only now prepared to tackle it.

Mike explains that to reach a pick-up point, the shortest distances will always require one to move a fixed number of units down (south) and a fixed number across (east) and observes that within the grid one cannot travel diagonally. Brian reminds the others that only going forward will produce a shortest route. Mike generalizes his awareness to all routes. Jeff signals that he is convinced, saying, "I mean, that's the most sensible way I think to say that." In the process of the group's discourse, Jeff and Mike help Romina to see that area is not an operative idea in this task.

In the above conversational exchange, the participants engage in socially emergent cognition (Powell, 2006), providing discursive evidence to several ideas: (1) movement within the given portion of the taxicab plane goes left or right and up or down; (2) diagonal movements are not permissible; (3) the taxi stand and each pick-up point together define a rectangle in which the pair of points are located at opposite ends of a diagonal, and the problem task involves moving along the perimeter but does not concern the extent of space that a rectangle occupies; (4) the number of units down plus the number of units across are objects related by addition to produce the length of a shortest path; (5) any route to the blue pick-up point will involve four blocks down and one block across ; and (6) each horizontal and vertical line segment of the grid can be considered as one unit in length.

By the end of the exchange, Jeff, who in the form of a question reintroduced Brian's proposal that they justify the idea that the length of efficient routes from the taxi stand to a pick-up point are equivalent, expresses satisfaction with Brian and Mike's argumentation ("I mean, that's the most sensible way I think to say that."), checks whether the others agree ("Right?"), and reminds his colleagues that they can now turn their attention of the crux of their task: "And they want to know how many though."

The discursive exchanges in the three episodes quoted above are critical. They present the major occasion in which the participants ferret out the nature of the problem space and build fundamental ideas essential for investigating the problem task. The participants establish what are the basic objects of taxicab geometry (points and line segments or routes); basic awareness of the Taxicab Problem (there can be more than one shortest route to a intersection point in the taxicab plane); and implicitly note a distinguishing feature between Euclidean and taxicab geometries (how distance is measured). This distinction emerges when Mike observes that in the context of the problem task, one cannot travel diagonally, he touches upon the fundamental distinction between the metric of Euclidean geometry and that of taxicab geometry. Moving forward with the ideas they have built that were illustrated in the three episodes, the students shift their focus to delve further into the problem task and generate considerably more data.

13.2.2.2 Generalizing to specialize

The students take a decisive turn in their investigation. They generalize the problem. Instead of determining the number of shortest paths between each of the three specialized pairs of endpoints, the work to uncover a pattern among the numerical values that represent the number of shortest paths between the taxi stand and any point on the grid. The first examine points in close proximity to the taxi stand. This decision proves to be a watershed event in their mathematical work on the Taxicab Problem.

13.2.2.3 Building isomorphisms to justify

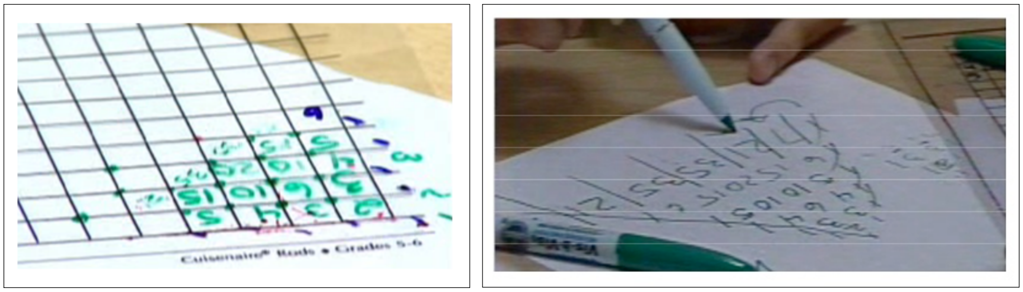
The transcription of the problem solving session contains 1869 turns of speech. The portion of the transcript that relates to the students building an isomorphism to justify their solution transpires over many turns of speech, spanning from turn 159 to turn 1320. Space does not permit us to present a full illustration of the development of the ideas and reasoning that comprise the students' work toward justifying their solution. They have continual discursive interactions with the aim of building an isomorphism between a rule for generating the entries of Pascal's Triangle and the number of shortest routes to points on the taxicab grid. Early in their work, they manifest embryonic thinking about an isomorphism. Romina wonders aloud: “can’t we do towers on this?” (This group’s previous work on the towers problem is discussed in Chapters 4, 5, 10, and 12.) Her public query catalyzes a negotiatory interlocution among Mike, Jeff, and her. Jeff, responding immediately to Romina, says, “that’s what I’m saying,” and invites her to think with him about the dyadic choice that one has at intersections of the taxicab grid. Furthermore, he wonders whether one can find the number of shortest routes to a pick-up point by adding up the different choices one encounters in route to the point. Romina proposes that since the length of a shortest route to the red pick-up point is 10, then “ten could be like the number of blocks we have in the tower.” Romina’s query concerning the application of towers to the present problem task prompts Mike’s engagement with the idea, as well. As if advising his colleagues and himself, he reacts in part by saying, “think of the possibilities of doing this and then doing that.” While uttering these words, he points at an intersection; from that intersection gestures first downward (“doing this”), returns the to point, and then motions rightward (“doing that”). Similar to Jeff’s words and gestures, Mike’s actions also acknowledge cognitively and corporally the dyadic-choice aspect of the problem task. Through their negotiatory interactions, Mike, Jeff, and Romina raised the prospect of as well as provided insights for building an isomorphism between the Taxicab and Towers Problems.

The prospect and work of building such an isomorphism reemerges several more times in the participants’ interlocution, and each time, they further elaborate their insights and advance more isomorphic propositions. Eventually, the building of isomorphisms dominates their conversational exchanges. Approximately thirty-five minutes after Romina first broached the possibility of relating attributes of the Towers Problem to the problem at hand, the participants reengage with the idea. Romina speculates that between the two problems one can relate “like lines over” to “like the color” and then “the lines down” to the “number of blocks.” What is essential here is Romina’s apparent awareness that each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem.

Romina uses this insight later in the session. She transfers the data that she and her colleagues have generated from a transparency of a 1-centimeter grid to plain paper. Their data are equivalent to binomial coefficients. She identifies one unit of horizontal distance with one Unifix cube of color *A* and one unit of vertical distance with one Unifix cube of color *B*:

> Like doesn’t the two- there’s- that I mean, that’s one- that means it’s one of *A* color, one of *B* color [pointing to the 2 in Pascal’s Triangle]. Here’s one- it’s either one- either way you go. It’s one of across and one down [pointing to a number on the transparency grid and motions with her pen to go across and down]. And for three that means there’s two *A* color and one *B* color [pointing to a 3 in Pascal’s Triangle], so here it’s two across, one down or the other way [tracing across and down on the transparency grid] you can get three is two down [pointing to the grid].

Furthering the building of their isomorphism, Mike offers another propositional foundation. Pointing at their data on the transparency grid and referring to its diagonals as rows, he notes that each row of the data refers to the number of shortest routes to particular points of a particular length. For instance, pointing the array—1 4 6 4 1—of their transparency, he observes that each number refers to an intersection point whose “shortest route is four.” Moreover, he remarks that one could name a diagonal by, for example, “six” since “everything [each intersection point] in the row [diagonal] has shortest route of six.” In terms of an isomorphism, Mike’s observation points in two different directions: (1) it relates diagonals of information in their data to rows of numbers in Pascal’s Triangle and (2) it notes that intersection points whose shortest routes have the same length can have different numbers of shortest routes.



A B

Figure 13-1. Participants' data arrays A and B

Later in responding to a researcher's question, the participants develop a proposition that relates how they know that a particular intersection in the taxicab grid corresponds to a number in Pascal's Triangle. They focus their attention on their inscriptions in Figure 13-1, which shows empirical data of shortest routes between the taxi stand and nearby intersection points. In array A, the green numbers (lighter shade inside boxes) show empirical data of shortest routes between the taxi stand and nearby intersection points. Jeff wrote the 1s on the side in blue (darker shade) to augment the appearance of the numerical array as Pascal's Triangle. From the participant perspective, to the left of Jeff's numbers, Romina wrote in green (ligher shade) the numbers 1, 2, and 3 to indicate the row numbers of the triangular array. Array B shows their drawing of Pascal's Triangle. The first five rows contain empirical data; the remaining two rows contain assumed data values based on the addition rule for Pascal's Triangle.

Mike and Romina discuss correspondences between the two inscriptions. Referring to a point on their grid that is five units east and two units south, Romina associates the length of its shortest route, which is seven, to a row of her Pascal's Triangle by counting down seven rows and saying, "five of one thing and two of another thing." Mike inquires about her meaning for "five and two." Both Romina and Brian respond, "five across and two down." She then associates the combinatorial numbers in the seventh row of her Pascal's Triangle to the idea of "five of one thing and two of another thing," specifying that, left to right from her perspective, the first 21 represents two of one color, while the second 21 "is five of one color," presuming the same color. Using this special case, Romina hints at a general proposition for an isomorphism between the Taxicab and Towers Problems.

13.3 Conclusion

The narrative of these four students working on the Taxicab Problem has three sections. The first concerns their recognition of the need to justify an observation that they made immediately after reading the problem statement. The observation maybe simple but their recognition of the significance of the observation and that it needed to be justified before progressing on with resolving the problem is rather sophisticated. This sophistication in their mathematical work speaks to the sociomathematical norms (Yackel & Cobb, 1996) that they have developed through their longitudinal experience working on open-ended problems. This sociomathematical norm is subtle and akin to the way mathematicians work.

The second section of the narrative pertains to their decision to seek a general solution to the problem and that such a solution would be easier than trying to count the number of shortest routes between each of the three pair of given endpoints. This is an instance of what can be called generalizing to specialize. That is, finding a general solution of a problem situation in order to answer more specific questions of the problem. Often the general case is easier to solve than special cases.

Finally, the third section of the narrative revolves around not only with the recognition that claims needs to be justified but also with a particular proof strategy that emerged in the students' attempt to justify their resolution of a generalized form of the Taxicab Problem.

Important sociomathematical norms (Yackel & Cobb, 1996) are evident in the students' mathematical interactions in the first and third threads: claims need to be justified and a problem's solution needs to be connected or linked to attributes of the problem. These norms emerge from the mathematical interactions of students who have a collective history of problem solving through occasional interactions over their school years with researchers from Rutgers University who increasingly over the years left the students to structure their own mathematical investigations in response to given tasks.

In this chapter and the previous chapters of this section, we have given the researchers' perspectives on the students' mathematical work. In Chapter 14, we examine this work from the point of view of the students.