Chapter 12: Representations and Standard Notation

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Tasks:	Towers, Pizzas, and Pascal's Triangle
Participants:	Ankur, Brian, Jeff, Mike, and Romina
Researchers:	Carolyn Maher and Regina Kiczek

12.1 Introduction

In the preceding chapters in this section, we considered how students made sense of Pascal's Triangle and isomorphic combinatorics problems using their own increasingly sophisticated and abstract representations. In this chapter, we see how one group built on those ideas in order to derive, explain, and record Pascal's Identity (the addition rule for Pascal's Triangle) using standard mathematical notation. This remarkable demonstration of how students can come to make sense of complex mathematical ideas was captured during the session that came to be referred to as the "Night Session," since it took place on a weekday evening from 7:30 to 10:00 PM.

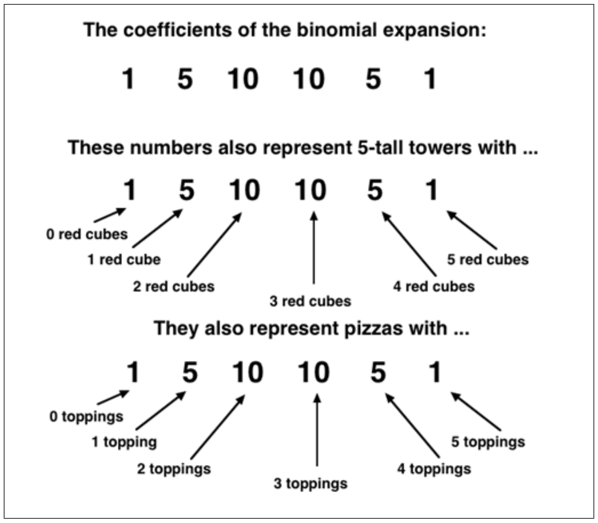
In Chapter 10, we described how, during their sophomore and junior years of high school, Ankur, Brian, Jeff, Mike, and Romina made use of the pizza and towers problems to develop complex mathematical notions: recognize isomorphisms, generalize findings, and represent ideas using their own personal representations, which had become increasingly sophisticated and symbolic over the years. In the session described in this chapter, they made use of standard combinatorial notation to communicate, clearly and concisely, the ideas about Pascal's Triangle and Pascal's Identity that they had previously developed. They derived Pascal's Identity, wrote it in standard notation, and explained the meaning of the standard notation in terms of general versions of the pizza and towers problems.

In the following sections we discuss the strategies used by the students to make sense of the standard notation. We note how the use of increasingly sophisticated notation accompanied the students' building of general notions about the meaning of Pascal's Identity. We show how their organizational strategies proved key in their making sense of the standard notation and of the relationships between the combinatorial problems, Pascal's Triangle, and Pascal's Identity. Further, we show that they found in the standard notation an essential tool for expressing their understanding of Pascal's Identity in general form.

12.2 Summary of Earlier Student Work

As discussed in Chapter 10, during sophomore- and junior-year problem-solving sessions, Ankur, Brian, Jeff, Mike, and Romina revisited and extended previous work on two familiar combinatorial problems. In the first session of their sophomore year of high school, they were reintroduced to the towers and pizza problems. In subsequent sessions, they found a way to organize their solution lists to prove that all solutions were present; they recognized that those problems were related to each other, to the binomial coefficients, and to Pascal's Triangle; they found general solutions to those problems; and they used those problems to form preliminary ideas about the meaning of Pascal's Identity.

The key organizational decision—to organize pizzas by number of toppings and towers by number of cubes of a given color—not only helped the students show that they had all the solutions, but it also helped them see the relationship between the two problems and Pascal's Triangle. It enabled the students to realize, for example, that the fifth row of Pascal's Triangle (1 5 10 10 5 1) contains not only the coefficients of the binomial expansion but also the solution to both the five-topping pizza problem and the five-tall towers problem. Refer to Figure 12-1.



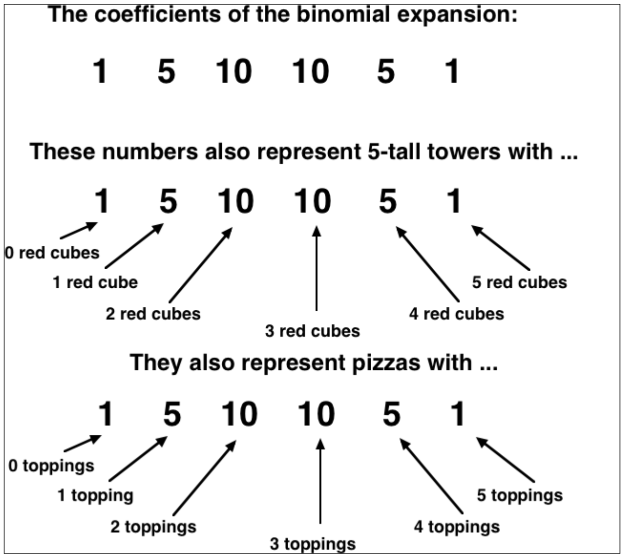


Figure 12-1. Row 5 of Pascal's Triangle

When the students were introduced to the use of standard notation to describe the binomial coefficients, they were able to make use of the connection they had already formed between the binomial coefficients and the towers and pizza problems. They applied the standard notation to the pizza and towers problems; knowing how to generate the answers to the pizza and towers problems in their own notation enabled them to use the standard notation to describe the general pizza and towers answers and finally to express the general rule for building Pascal's Triangle.

12.3 The Night Session

In the Night Session Ankur, Brian, Jeff, Mike, and Romina returned to the investigations of Pascal's Triangle that they had begun in their sophomore year. By the end of this session, they had written Pascal's Identity in standard notation and provided a sound explanation of its meaning. They did this by looking at general forms of the towers and pizza problems, referring back to their previous explanations of specific instances of Pascal's Identity in terms of towers and pizzas, and making use of the binary notation that had been introduced by Mike some 18 months earlier.

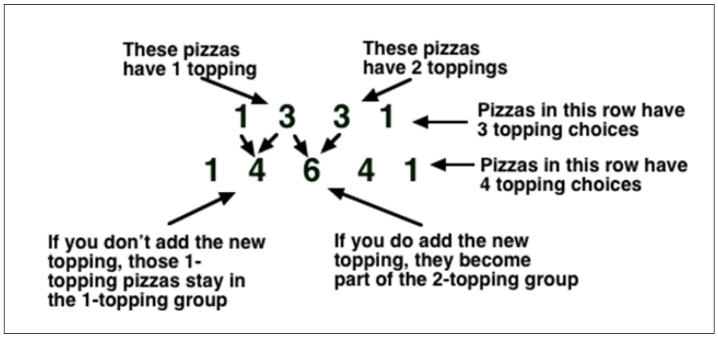
At the beginning of the session, the first three students to arrive (Jeff, Mike, and Romina) talked about that day's class work, which had been to find the coefficients of the expansion of $(a+b)^n$. Jeff brought up what they called "choose" notation, the notation to denote combinations, using the ${}_nC_r$ function on their calculators.

In this episode, when Jeff was trying to explain how to find a particular coefficient of the expansion of $(a+b)^{10}$, Romina spontaneously introduced the towers problem with the words "ten high" and "two reds." Jeff and Mike elaborated that this meant building towers ten cubes tall, selecting from two colors, and counting how many there are containing exactly two cubes of one of the colors (red).

JEFF:	… If we were looking for *a* plus *b* to the tenth say, … it was 1 *a* to the tenth and then 10 *a* to the ninth *b* to the first, right? … [The next coefficient] was 45, but we were working on how to figure it out. We knew it was the choose thing, whatever that means. … What was it? Ten choose two? … Like, uh, was it N-C-R? [Jeff is referring to buttons on his calculator.] Two, is that how you do it? [Jeff writes 10 nCr 2.] Right? … And that equals 45, and that's the answer. … We're not really sure how all this works but it's like … If you have ten different. What is it? Ten different things …
ROMINA:	Ten high. Ten high.
JEFF:	Ten high. How many.
ROMINA:	How many would have two reds, only two reds.
JEFF:	How many would have two, two reds.
RESEARCHER:	One more time.
JEFF:	If you have towers with ten high and two colors.
MIKE:	How many different places can you put two reds in there?
JEFF:	And like *a* would be one color and *b* would be the other color.

Their original explanation of "ten choose two" was so brief, it would have been difficult for anyone not familiar with their work to understand the references "ten high" and "how many would have two reds." But the elaboration (although it still assumed knowledge of the towers problem) shows that they knew that the coefficients of the binomial expansion to the tenth power were related to the 10-tall towers problem.

A few minutes later when the researcher asked the group to discuss the "choose" notation that Jeff had mentioned, Mike drew a few rows of Pascal's Triangle on the board and explained that any row could be expressed in "choose" notation; for example, the row 1 3 3 1 could be called "3 choose 0" through "3 choose 3." When the researcher asked the students to talk about the addition rule for Pascal's Triangle in that notation, Romina suggested a new vehicle, the pizza problem, even though she had just used the towers problem in the previous explanation. Mike used the pizza problem to explain a particular case of addition: think of the n^{th} row of Pascal's Triangle as representing all the possible pizzas that can be made when there are *n* toppings to choose from, and think of generating new rows of Pascal's Triangle as making new pizza toppings available. Then the pizzas represented by the first 3 in row 3 are the 1-topping pizzas (when there are 3 toppings to choose from). You can either add the new topping to those 3 pizzas (making them two-topping pizzas), or let them remain 1-topping pizzas. If you add the new topping, you now group them with the second 3 in that row (the pizzas that already have two toppings), resulting in 6 pizzas with two toppings. If you do not add the new topping, those 3 pizzas are added to the 1 pizza that had no toppings (and that had the new topping added to it), giving 4 pizzas with one topping. Figure 12-2 illustrates Mike's explanation. A portion of their discussion is given below.



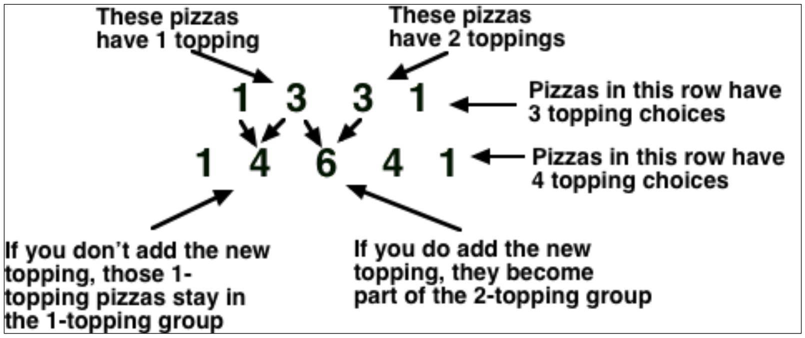


Figure 12-2. Examples of Pascal's Identity

MIKE: Let's go to this one. This would be like three different places, I guess. [Mike indicates row 3, which is 1 3 3 1.] …

JEFF: That would be *a* plus *b* to the third.

MIKE: All right, let's say you have like, here's a number, all right? [Mike writes 000.] Zero means no toppings. One would be a topping. So first category is everything with no toppings. [Mike points to the first 1 in row 3.] And that's your number for that one. [Mike points to 000.] That's like, like binary numbers or something. Next would be- [Mike writes 001, 010, then 101.] There's all the, the ones that have one topping.

JEFF: Right, you got to write that 0 at the end. You messed up. Last one should be a hundred, not a hundred and one.

MIKE: I knew that. [Mike changes 101 to 100.] There's all the ones that have one topping. … There's your 3 choose 1 and there's three different combinations you could put that. … But, um, when you have a new – when you add another place, another topping. [Mike draws dashes to the right of the four numbers already there. Refer to Figure 12-3.]

JEFF: That could be one or the other, one or the other, one or the other.

MIKE: So, it could be one or the other. It could be a zero or one, a zero or one, zero or one. [Mike writes 0 and 1 above each dash.] So all these threes would either move up a step onto the next category and have two toppings. [Mike points to the 6 in row 4.] Or they might stay behind and still only have one if they have the zero. [Mike points to the 4 in row 4.] So three get a topping, go to this one [Mike points to 6.] and three won't, will stay. [Mike points to 4.] These three [Mike points to the first 3 in row 3.] with one topping won't get one so, you know, you can put them in the same category as this one.

JEFF: That's their four? Yeah.

MIKE: That's four. … And you know, the three that had two toppings won't get any. [Mike draws a line from the second 3 in row 3 to the 6 in row 4. Refer to Figure

12-2.] And you could put them in together with the ones that did get something. That's why you would add.

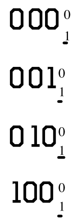


Figure 12-3. Binary listing of 3 choose 0 and 3 choose 1

After Mike explained the specific instances of 3+3=6 and 1+3=4 in terms of pizzas, the researcher (R1) rewrote row 3 of Pascal's Triangle in standard combinatorial notation and asked the students to write other rows in that form and show an example of the addition rule. Figure 12-4 shows what they did. Their discussion follows.

RESEARCHER: Show me that 3 plus 3 is 6. Which ones would it be? …
MIKE: This one and that one. [Mike points from 3 choose 1 and 3 choose 2 to 4 choose 2.] …
RESEARCHER: Okay, so you're saying 3 choose 1 plus 3 choose 2 equals 4 choose 2. Right? Okay. So what's 4 choose 2 plus 4 choose 3?
JEFF: … 4 choose 2 plus 4 choose 3? That would be, that would be 5-
ANKUR: 5 choose. …
MIKE: 5 choose 3.
JEFF: Why is he 5 choose 3?
ANKUR: Because it's always the one on the right. [Ankur means that the "choose" number of the sum is the same as the "choose" number of the rightmost addend.]

Mike observed that the bottom number indicated the number of toppings actually used, so that when a topping was added, the bottom number changed and when a topping was not added, the bottom number did not change.

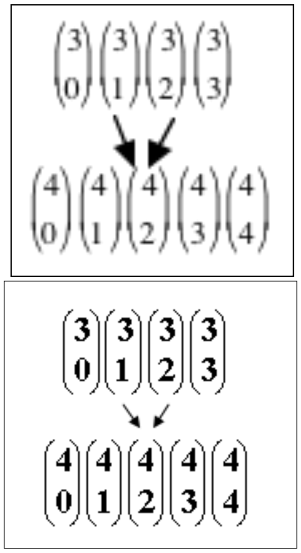


Figure 12-4. Showing 3+3=6 in combinatorial notation

The researcher asked the group to continue by writing a general (n^{th}) row of Pascal's Triangle and to use that row to discuss the meaning of the addition rule. Figure 12-5 shows the two general rows that Jeff drew.

In spite of the researcher's suggestion to use lower-case *n* to indicate row number and *r* to indicate a number in the middle of the row (following standard usage), Jeff used upper-case letters *N* and *X*.

$$
\begin{array}{c}
\binom{N-1}{0} \cdots \binom{N-1}{X} \cdots \binom{N-1}{N-1} \\
\binom{N}{0} \cdots \binom{N}{X-2} \binom{N}{X-1} \binom{N}{X} \binom{N}{X+1} \binom{N}{X+2} \cdots \binom{N}{N}
\end{array}
$$

Figure 12-5. Rows *N*-1 and *N* of Pascal's Triangle

Brian arrived after the group had been working for almost an hour; first Jeff explained to Brian how Figure 12-5 related to that day's work in their regular math class relating to Pascal's Triangle:

> We're explaining the general addition, the addition rule using chooses to fill out the triangle, and this here would be N choose X plus 1 and then N choose X plus 2 and so on to whatever N equals.

Then the group was asked to write the addition rule in general form. Figure 12-7 shows Jeff working at the board as they discussed the problem. This discussion follows.

RESEARCHER:	Can you write it as an equation? Just like you wrote three plus three equals six.
ANKUR:	*N* plus, just that plus that. [Ankur points to the entries *N* choose *X* and *N* choose *X* + 1.]
MIKE:	*N* choose *X*.
JEFF:	*N* choose *X* plus *N* choose *X* plus one. [Jeff writes on the board as he speaks. Refer to Figure 12-5.]
MIKE:	Equals that. …
JEFF:	Plus one equals that right there. [Jeff points to *N*+1 choose *X*+1.] … Then, well, that's, that's because this would be gaining an *X* and going into the *X* plus 1. [Jeff points to *N* choose *X*.]
MIKE:	Yeah.
JEFF:	And this would be losing an *X*. [Jeff points to *N* choose *X*+1.]
MIKE:	No, no, not losing, not getting anything.
ANKUR:	Staying the same.
MIKE:	And the top numbers have changed because you have more.
JEFF:	Because you're adding; you have more things [to choose from].
RESEARCHER:	Say it so Brian can follow it because he wasn't here for the earlier pizza discussion.
JEFF:	What, what we're doing is the next line of the triangle. Remember how today in class the other triangle was one, two.
BRIAN:	Yeah.
JEFF:	Three, that whole row there. Well, that's the increase in *N* and then the *X* plus one. … Say we're doing pizzas.
BRIAN:	All right.
JEFF:	If you add another topping onto it.
ROMINA:	You know how we get the triangle and how we go 1 2 1 and add those two together?
BRIAN:	Yeah.
JEFF:	We were explaining why you add.
BRIAN:	All right, keep going.
JEFF:	Because [Jeff points to *N* choose *X*+1.] … If it gets a topping, that's why it goes up to the *X* plus 1. [When a new topping is available, the second ("choose") number in the expression is increased by 1.] And in this one, it's staying the

same, right? [Jeff points to *N*+1 choose *X*.] And that's why it's going there. Make sense?

BRIAN: Yes. It actually does.

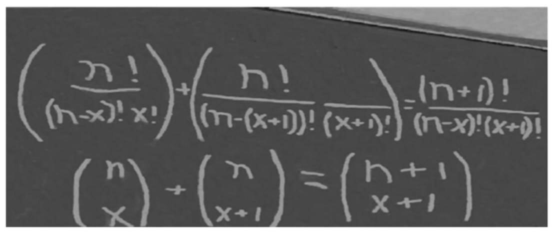
JEFF: So, so that would be the general addition rule in this case.

$$\binom{N}{X}+\binom{N}{X+1}=\binom{N+1}{X+1} \qquad \binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$$

Figure 12-6. Pascal's Identity in students' notation and as shown in textbooks

Figure 12-7. Jeff works on Pascal's Identity

The students' version of Pascal's Identity is equivalent to a standard textbook version of this equation, with *n* equivalent to *N*+1 and *r* equivalent to *X*+1. Following the production of the equation in combinatorics notation, the students were asked to convert that notation to factorial notation. They did so; their work is shown in Figure 12-8.



$$\left(\frac{n!}{(n-x)!\,x!}\right)+\left(\frac{n!}{(n-(x+1))!\,(x+1)!}\right)=\frac{(n+1)!}{(n-x)!(x+1)!}$$

$$\binom{n}{x}+\binom{n}{x+1}=\binom{n+1}{x+1}$$

Figure 12-8. Two versions of Pascal's Identity

Earlier in this session, Mike had explained 3+3=6 (Figure 12-3) by stating that the 3's are from the three-toppings row of Pascal's Triangle: the first 3 represents the three 1-topping pizzas that become two-topping pizzas, and the second 3 represents the three two-topping pizzas that remain two-topping pizzas; the 6 represents the six two-topping pizzas that can be made when there is a fourth topping available. Now the students generalized this rule using the standard notation, which they called "choose" notation: *N* choose *X* gives the number of pizzas that have exactly *X* toppings when there are *N* toppings to select from and *N* choose *X*+1 gives the number of pizzas that have exactly *X*+1 toppings. Moving to the next row down in Pascal's Triangle means that you increase the number of available toppings by 1, and so *N* increases by 1. Adding the new topping to the first addend (*N* choose *X*) and not adding the new topping to the second addend (*N* choose *X*+1) gives a group of pizzas with *X*+1 toppings when there are *N*+1 toppings to select from.

The students had described instances of Pascal's Identity in earlier sessions. For example, in their sophomore year, Ankur, Jeff, and Romina had provided a specific explanation using towers: In order to create a six-tall tower with exactly 3 red cubes, add a red cube to the five-tall towers that have exactly 2 red cubes and add a blue cube to the five-tall towers that have already have 3 red cubes. (This was discussed in more detail in Chapter 10.) But this Night Session explanation was the first explanation of Pascal’s Identity using standard notation to state a general result.

12.4 Durability of Understanding

Three years after the Night Session, in individual interviews, Mike, Romina, and Ankur were asked to recall this work. In that time, all three took math classes in college, but none studied combinatorics. All three were able when prompted to write the formula, and all three were able to provide a cogent explanation of the addition rule. As an example, we discuss here the interview with Mike.

Over the years in which he worked on the combinatorial problems, Mike progressed from drawings and codes through his personal (binary) representation system to the standard combinatorial notation. From the time he introduced his ideas about binary notation to his fellow students to his most recent interview over five years later, he demonstrated the ability to make sense of the problems and of the notation, both through the use of his chosen notation and through the use of the combinatorics tasks. Mike took the lead in devising representations, finding connections, and making sense of the tasks. He recognized structural similarities between problems; he moved between different representations with ease; and he extended, generalized, and reorganized his knowledge when he discussed it with others.

In this interview, which took place when Mike was in his second year of college, the researcher (R1) showed Mike a diagram of Pascal’s Triangle and asked him to recall how the group used the pizza problem to think about Pascal’s Triangle. Mike spent a little time regenerating the meaning of specific entries in Pascal’s Triangle and then, in response to a question about a general rule, he reconstructed the formula that had been developed during the Night Session. A portion of their discussion follows. Mike began with the two-topping pizza problem.

MIKE: Okay. If you had no toppings, that would be one pizza.

RESEARCHER: Okay. So where is that on the triangle?

MIKE: Well, I'm going to just draw it. … And then we'll find it. … If you're using just one topping, you can make two possible pizzas with that. And then if you have all the toppings, that's one. Right. And then automatically I see that relates to this row. [Mike points to row 2 of Pascal’s Triangle (1 2 1).] And I'm pretty sure it would go down, this is like a third topping and a fourth topping. [Mike points to rows 3 and 4 of Pascal’s Triangle.] Now I think the way I thought about it is, like, the row on the outside [leftmost entry in a given row] would be your plain pizza. And there's only one way to make a plain pizza. And … the next one over would be how many pizzas you could make using only one topping, and then so on until you get to the last row [the rightmost entry in a given row] which is all your toppings. And, once again, you can only make one pizza out of that. …

The researcher then suggested that Mike work on a general rule:

RESEARCHER: And at that session, I asked them to write an equation to show, for instance, how that might happen from one row to the next. So can you just do that, write. …

MIKE: Like a general equation?

RESEARCHER: Well, that was what I was going for ultimately. …

MIKE: To give an amount for any spot in the row.

RESEARCHER: Right. …

MIKE: All right, so I guess we'll give, you know, the row a name. Call that *r*. And I guess the spot in the row, like, you know, zero topping, one topping. Call that, *n* sounds fine. [There is a pause; then Mike writes the left part of the equation shown in Figure 12-8.] I'm just going to like work this out in my head and see if it actually works. [A few seconds later, Mike adds the right part of the equation.]

This equation, shown in Figure 12-8, is equivalent to the textbook version and to the Night Session equation, although he used different variables. (Textbooks usually use "*n* choose *r*" instead of "*r* choose *n*," and the sum is given on the right side). We can see that Mike did not rely on symbol manipulation. He linked the numbers to a problem task that made sense to him, and then he expressed the relationships in that task in symbolic form. We conclude that Mike was reconstructing the substance rather than merely remembering the form.

$$\binom{r}{n} + \binom{r}{n+1} = \binom{r+1}{n+1}$$

Figure 12-8. Mike's equation for Pascal's Identity

12.5 Discussion

Exploring previously unexamined complexities of the towers and pizzas problems was a mathematically challenging task of the sort recommended by Davis and Maher (1990) to foster students' ability to engage in real mathematics – developing their own mathematical theories, for example. Conditions important for the development of new mathematical ideas were in place: these students had ample time for exploration of mathematical ideas and the opportunity to express their own ideas. The students' existing representations were taxed by new questions about how to relate these problems to each other, to Pascal's Triangle, and to the binomial coefficients and about how to represent a general instance of Pascal's Identity. Hence, there was a need to reorganize existing knowledge and to use new tools for dealing with these new ideas. We have shown that these students did make use of a new tool – standard mathematical notation – for dealing with their ideas about Pascal's Identity.

When they first started working on the pizza and towers problems, Ankur, Brian, Jeff, Mike, and Romina built towers and drew pictures of pizzas. As early as middle school, they began instead to use symbolic notation. (For example, they used letter and number codes to stand for the objects they were investigating.) Besides continuing the use of codes during high school, the students also found increasing use for the standard notations of mathematical discourse. For example, the binary notation that they began to use in high school was more powerful than the letter codes because it was easily extended (adding a cube to the tower or a topping choice to a pizza corresponded to adding a binary digit) and it was applicable to both pizza and towers problems, thus making it easier for the students to identify the similar structures of the two problems. Using binary notation helped the students focus on the isomorphic structural aspects of the combinatorial problems (the duality of the choices) rather than the surface features (the different pizza toppings, for example). Binary notation was also an easily-generalizable notation in three ways: 1) adding an extra digit corresponded to making a new pizza topping available and to increasing the tower's height by one block; 2) adding a 1 corresponded to adding that newly available topping and to adding a block of the designated color to the tower; and 3) adding a 0 corresponded to not adding the new topping and adding a block of the other color. This idea that it was not necessary to know the current number (or names) of pizza toppings or the current height of the tower in order to describe what happened next was important in the students' production of the general equation for Pascal's Identity.

During the course of discussions over four months of their sophomore year, these students first noted that the pizza and towers problems had the same answer in specific cases. Then they linked specific answers to the pizza and towers problems to specific entries in Pascal's Triangle. Finally, they described the links among binomial coefficients, pizza toppings, and towers. (Blue block = a = topping on the pizza; white block = b = topping off the pizza.) During the Night Session, they built on their knowledge of these links in order to produce the general form of Pascal's Identity. We claim that their ability to map corresponding mathematical structures among these three representations is a strong indication of their mathematical competency and it indicates more competency than, for example, simply being able to reproduce or use a memorized formula.

The way these students organized their answers to the pizza and towers problems was a key organizational element that helped them to form connections among those problems and Pascal's Triangle.

They also made extensive use of their personal representations at the beginning of the process. But once those connections were formed, the students began to make general statements about Pascal's Triangle and Pascal's Identity, and they had less use for personal representations. Finally, although they were able to articulate general information about Pascal's Triangle and Pascal's Identity, they did not represent the generalizations symbolically until the Night Session.

After they had made the association between Pascal's Triangle and the combinatorial problems, the students demonstrated an ability to describe any selected entries in Pascal's Triangle in terms of the combinatorial problems. For example, they described the numbers in row 6 as representing six-tall towers with zero through six red cubes respectively. The fact that they could explain any instance suggested that they had an idea of the general rule; but without the standard notation, they could express their general ideas most easily by referring to specific examples. By the time of the Night Session, these students seemed to know general rules about generating Pascal's Triangle, but they lacked the notation to express these rules in a concise way. They were at the point where they needed standard notation in order to proceed further.

We suggest that these findings point to one way that teachers can follow the recommendation by the NCTM (2000) to "use sound professional judgment when deciding when and how to help students move toward conventional representation" (p. 284). Teachers should aim to help students to develop a powerful organization, one that lends itself to a mapping onto formal notation. In that way, the formal notation can be seen as the solution to a problem that arises during the students' own investigations: the problem of how to express in a general way the findings that the students have developed on their own.

In this chapter, we have seen how this group of students learned about the relationships among well-known combinatorics problems and Pascal's Triangle. In the following chapter, we observe the same students working on a new problems in combinatorics and using what they learned about the pizza and towers problems and their relation to Pascal's Triangle in order to make sense of that unfamiliar problem – the Taxicab Problem.