Chapter 10: Representations and Connections

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Dates and Grade:	December 1997 through March 1998; Grade 10
Tasks:	Towers and pizzas
Participants:	Ankur, Brian, Jeff, Mike, and Romina
Researchers:	Carolyn Maher and Robert Speiser

10.1 Introduction

In the previous chapter, we viewed a cohort of high school students from the longitudinal study as they explored the towers problems. In this chapter, we observe a different cohort of students also exploring the towers and pizza problems. In the five sessions discussed here, spanning December 1997 through March 1998, the five students in this cohort were reintroduced to the towers and pizza problems, which they had last seen in elementary school. (Refer to Chapters 5 and 6.) They found general solutions to those problems; they found a way to organize their solution lists to prove that all solutions were present; they recognized that those problems were related to each other, to the binomial coefficients, and to Pascal's Triangle; and they used those problems to form preliminary ideas about the meaning of Pascal's Identity. We show how their development and use of a sophisticated general representation scheme helped them make these connections and generalize their knowledge.

10.2 Session 1: A Common Notation

On December 12, 1997, when they were in the tenth grade, Ankur, Brian, Jeff, Mike, and Romina began to meet with the researchers as a group in one- to two-hour after school sessions that continued throughout high school. At their first meeting, the researchers proposed the pizza problem to them as they snacked – on pizza. When first presented with the four-topping problem in the fourth grade, the students had enthusiastically tackled the problem by randomly generating combinations. In approximately 15 minutes they had found all 16 possibilities, using an alphabetical coding scheme to represent each pizza as it was generated. (Refer to Chapter 5 for details.)

In tenth grade, working on the three-, four-, and five-topping pizza problem, Jeff and Romina utilized an alphabetical coding scheme similar to the one that they used in fourth grade (*p* for pepperoni, *m* for mushroom, etc.). Ankur and Brian used a numerical coding scheme (1 through *n* for the *n* different toppings). Mike, however, worked alone. He proceeded on an unexpected path, choosing a unique binary number coding scheme, which became a fixture in many of their future discussions of combinatorial problems.

As the first four students discussed the problem, they realized that they needed a common notation; they adopted the alphabetical model. They kept the plain pizza separated, and so they found 7 possible pizzas for the three-topping case (plus plain) and 15 pizzas for the four-topping case (plus plain). When they found 30 five-topping pizzas (plus plain), they realized it did not fit the pattern. They hypothesized that a doubling rule might be involved and they decided to rethink their solution. At this point, Mike re-entered the discussion in order to introduce his binary coding scheme. He proposed that pizzas be represented by binary numbers; a four-topping pizza would be represented by a four-digit binary number, with a 1 in the k^{th} digit representing the presence of the k^{th} topping and a 0 representing the absence of the k^{th} topping. For example, all one-topping pizzas are represented by all four-digit binary numbers with exactly one 1: 0001, 0010, 0100, and 1000. This coding scheme was more easily generalizable than the letter code scheme: to add another topping, just add another binary digit. After listening to Mike's explanation of the binary code, the group made a connection between the two representations (letter codes and binary notation: use the letters that stand for toppings as column headers for the list of binary digits. Refer to Figure 10-1 for their table; O stands for onion, M for mushroom, P for pepperoni, and S for sausage.

8	**4**	**2**	**1**	
O	**M**	**P**	**S**	
1	0	0	0	← onion pizza
0	1	0	0	← mushroom pizza

Figure 10-1. Students' table linking topping codes and binary notation [annotation added]

Mike's understanding of the binary system and the way it could be used to describe the solution to the pizza problem gave him an insight into a generalization about the number of pizzas that can be created from n toppings; Mike hypothesized that the answer to the n-topping pizza problem is 2^n. The group discussed the numerical solution for some time – there was some confusion about whether the coefficient might be $n - 1$ or $n + 1$, or whether the answer might be $2^n - 1$, possibly due to the uncertainty about how to count the plain pizza, but ultimately all agreed that the solution was 2^n.

As they were wrapping up the session, the researcher asked if the problem reminded them of anything, and Brian mentioned towers: "Every thing we ever do always is like the tower problem." In order to investigate the possible relationship between the two problems, the students worked on the three-tall tower problem and concluded that the answer was the same as for the three-topping pizza problem; there are eight three-tall towers, just as there are eight possible three-topping pizzas. Because they were focused on relating the pizza toppings to the cubes' colors, they concluded that the problems were similar but not identical. Ankur noted for example that a red-yellow tower is different from a yellow-red tower, but a pepper-pepperoni pizza is the same as a pepperoni-pepper pizza.

10.3 Session 2: Towers and Pizzas

One week later, the students returned and resumed their discussion of a possible relationship between pizza and towers problems. Although they were asked to consider only the two-color towers problems, they kept returning to the question of how to count the possible number of towers when there were cubes of three or more colors. Looking at this issue led them to the realization that when the height of the towers is the only variable under consideration, the towers problem is identical to the pizza problem. This time, they mapped the height of the tower to the number of pizza topping choices (contrary to the previous week, when they attempted to map number of colors to number of pizza topping choices). A portion of their discussion follows.

JEFF:	If the only variable we're changing is height, it stays the same.
MIKE:	It would be the same as the pizza.
JEFF:	What would that be like changing on the pizza, though?
MIKE:	You could change the height, the number of toppings.
JEFF:	Changing the height would be like changing the number of toppings.
ANKUR:	Yes.
JEFF:	Changing the color would be like, what?
ANKUR:	Say what you just said again.

JEFF: All right. When we change the height of the box, from like two to three, it's like changing the topping on the pizza from a possible two toppings to three toppings.
ROMINA: Okay.
ANKUR: Okay.

Brian and Mike returned to the question of changing the number of colors available for building towers, and Mike proposed that when there are three colors to choose from, there are nine possible two-tall towers (3 times 3). After a 10-minute discussion, the other four students agreed. In the course of the discussion, they attempted to clarify the meaning of the base and exponent in each problem. In doing this, they were able to answer a question that they had not answered in the previous session, that of the meaning of the 2 in the formula 2^n. In the previous session, they had determined that *n* represented the number of toppings, and in this session, they determined that the 2 stands for the two toppings choices: on or off the pizza:

BRIAN: Two has to stand for something.
MIKE: It stands for something; *n* was the number of toppings and 2 is what – you could either have O or 1. You either have a topping or not.

Later, Romina presented the group's findings to the researchers:

ROMINA: Okay. For the pizza problem, the 2 *n* [meaning 2^n], the two represents either topping or no topping. Right?
MIKE: There's two different possibilities for each.
JEFF: That's why there's two. We didn't know, I don't think we explained that last time, why it was two.
JEFF: Topping or no topping, and that's what the two is. Now the *n*, Romina.
ROMINA: Is toppings.
JEFF: The number of toppings.

Mike used binary notation again in this session, this time using it to represent the two colors of the towers problem. He noted that instead of relating binary digits to the presence or absence of pizza toppings, he could relate them to colors of cubes: "Zero is blue and one is red." Figure 10-2 shows Mike's table of solutions for both the two-topping pizza problem and the two-tall towers problem. The column headers 1 and 2 represent the two pizza topping choices and the two levels of the tower. The 1 and 0 represent topping/no topping and blue cube/red cube.

2	1		p	m
1	0		1	0
0	1		0	1
1	1		1	1
0	0		0	0

Figure 10-2. Mike's listing of two-tall towers and two-topping pizzas

Mike explained that if the labels at the top of the chart stood for pizza toppings (*m* for mushrooms and *p* for pepperoni), the zeros and ones would represent the presence or absence of the topping. If the labels stood for positions in towers, the zeros and ones would represent the color of a cube; e.g. one would represent blue and zero would represent red.

During the second half of this session, at the researcher's request, the students explored geometric interpretations of the binomial expansion. Figure 10-3 represents their drawing of $(a+b)^2$. They went on to spend over half an hour working on drawings and three-dimensional models for $(a+b)^3$, although no model was entirely satisfactory to them. These investigations can be seen as preparation for their later work describing the isomorphic relationship among the towers and pizza problems and the binomial expansion.

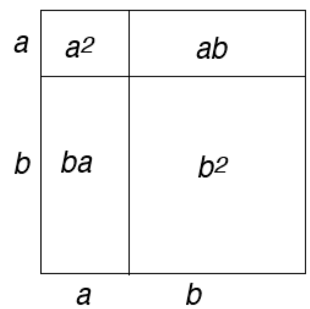

Figure 10-3. Geometric interpretation of $(a+b)^2$

In this session, the students gave the researchers clear and convincing explanations of the isomorphism between the pizza and towers problems and of the meaning of the components of the formula, and it appeared that they had a firm grasp of the underlying structure of the problems. But a few months later (in sessions discussed later in this chapter), we see them return to their focus on relating number of pizza toppings to number of colors (instead of to the height of the towers), once again deciding that the problems were similar but not identical. This illustrates how important it is to revisit problems and re-examine solutions, to solidify and expand students' understanding.

10.4 Session 3: Towers and the Binomial Expansion

When the students met in January 1998 after the holiday break, they returned to the topic of towers. The researchers gave them a problem from fourth grade: When you are choosing from red and yellow cubes, how many five-tall towers can you build containing exactly two red cubes? They immediately answered "ten," and they were then challenged to provide an explanation. Mike and Ankur provided a justification in approximately two minutes, using Mike's binary coding scheme, with 0 representing a yellow block and 1 representing a red block. As Ankur explained their solution, their organization improved; they begin to control for variables by holding the red cube fixed in the top position and then moving the second red block into successively lower positions until it reached the bottom position. This process was repeated, holding the red cube fixed in the second and then third and fourth positions. Figure 10-4 shows their original list followed by the re-organized list of all ten towers. This illustrates the importance of revisiting and re-explaining answers; in the process of explaining their solution, they organized the list so as to make it clear that all possibilities were accounted for and none were missing.

1	0	0	0	1	1	1	0	0	0
1	1	0	0	0	0	0	1	0	1
0	1	1	0	1	0	0	0	1	0
0	0	1	1	0	1	0	1	0	0
0	0	0	1	0	0	1	0	1	1

1	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	1	0	0	0
0	1	0	0	1	0	0	1	1	0
0	0	1	0	0	1	0	1	0	1
0	0	0	1	0	0	1	0	1	1

Figure 10-4. Mike and Ankur's two lists of five-tall towers with exactly two red cubes

The other group (Jeff, Romina, and Brian) worked on their solution for approximately 25 minutes. Their solution depended on first finding all possible five-tall towers; they recalled from previous work that there are 32 such towers. They built a justification based on cases; their cases were (1) all red: 1 tower, (2) one red and four yellow: 5 towers, (3) two red and three yellow: 10 towers, (4) three red and two yellow: 10 towers, (5) four red and one yellow: 5 towers, and (6) all yellow: 1 tower.

While Brian, Jeff, and Romina were working on their solution, and Ankur and Mike were done, Ankur proposed a problem that became known as Ankur's Challenge. The group's work on this problem was discussed in Chapter 8.

At the end of this session, the researcher introduced some of the notations of combinatorics. She told the students that asking how many five-tall towers have exactly two red cubes is the same as asking how many combinations there are when selecting two of five objects. She showed four different ways to write this, as shown in Figure 10-5.

$$ {}_5C_2 \qquad C_{(5,2)} \qquad \binom{5}{2} \qquad C_2^5 $$

Figure 10-5. Notation for selecting two of five objects

She concluded with a discussion of the binomial expansion and Pascal's Triangle. Following up on the previous session's investigation of the binomial expansion, she wrote the expansion of (a+b) to powers 0 through 3, drew Pascal's Triangle, and then asked the students to think about the relationship. (Refer to Figure 10-6 for the researcher's drawings).

$(a+b)^0 = 1$	1
$(a+b)^1 = 1a + 1b$	1 1
$(a+b)^2 = 1a^2 + 2ab + 1b^2$	1 2 1
$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1

Figure 10-6. The binomial expansion and Pascal's Triangle

In the following excerpt, the researcher hinted about the relationships that the students were in the process of discovering.

RESEARCHER:	The question is, what's the relationship here? How could you model it? How could you show this relationship? And why does it work? That's the question. So that's sort of the direction. Are you interested in knowing that? I think you have the bits and pieces to put it together.
ANKUR:	Some of the pieces are really small.
RESEARCHER:	They're bigger than you think. You've been working on this for a long time.
ROMINA:	Is this what we did today though?
RESEARCHER:	You've been dealing with some of this today. So think about it.
ANKUR:	So are all of the things we learned for the past eight years sort of combined into one thing?
BRIAN:	Imagine that.

Immediately following that discussion, the researcher asked the students to make concrete the numbers in Pascal's Triangle, by thinking about them in a "very real way" (linking them to towers problems).

RESEARCHER:	When you first came in here today, you produced that number ten. [She refers to the first 10 in row 5 of Pascal's Triangle – 1 5 10 10 5 1.] Right?
ANKUR:	Yes.
RESEARCHER:	And what problem were you solving?
ANKUR:	Two were red and three something else.
RESEARCHER:	Okay. So you can think of that ten in a very real way, if you want to, right?
ANKUR:	Yeah.
RESEARCHER:	Can you think of those other numbers in a real way? Does that help?
ANKUR:	The 1 is, in 1 4 6 4 1, the 1 represents all red. The other 1 represents all yellow I guess …
RESEARCHER:	All red and all yellow for what?
ANKUR:	Of four high.
RESEARCHER:	So this is four high. [The researcher points to row 4 of Pascal's Triangle.] And these are all red. [The researcher points to the first 1 in that row.]

This marks the first time the towers problem was explicitly linked with Pascal's Triangle, when row 4 of Pascal's Triangle was connected to the four-tall towers problem. Before the session ended, the researcher asked the students to think about the meaning behind the addition rule for Pascal's Triangle in the specific case of how the 6 in row 4 was generated from the two 3's in row 3. Although the students did not offer an answer at this session, it is noted here as the first time they were asked to think about Pascal's Identity. In this session we see three instances where the students were invited to think about how all the individual problems could be related, both to each other and to abstract mathematical entities, but without any explicit instruction about how to make a connection between the problems.

10.5 Session 4: Pizzas, Towers, and Pascal's Triangle

In this session, three students (Ankur, Jeff, and Romina), in a first meeting with visiting researcher Robert Speiser, explored the relationships among the pizza problem, the towers problem, and Pascal's Triangle, and (for the first time) they discussed Pascal's Identity in terms of operations on physical objects (adding cubes to towers).

When Professor Speiser asked the students about their recent work, the students did not mention a relationship between the towers and pizza problems. Instead, as they had initially done back in December, they maintained that the problems were different. Romina, recalling Ankur's earlier reasoning, said that red-blue cubes on a tower are different from blue-red cubes, but sausage-pepperoni is the same as pepperoni-sausage. Ankur added that a five-topping pizza problem is like a five-color towers problem. Jeff agreed, saying that a tower could have two of the same color but that a pizza could not have pepperoni-

pepperoni. Although all three had participated in the earlier, correct, discussion of the relationship between the towers and pizza problems, they recalled now only their own original ideas. They said nothing about the height of the tower being connected to the number of toppings, or how on-the-pizza/not-on-the-pizza could be made to correspond to blue/red cubes via the binary representations 0 and 1.

The researcher reminded the group about the combinatorics notation that had been introduced a month earlier and she reminded them how the notation was related to the five-tall towers problem. She went on to demonstrate the binomial expansion and to ask explicit questions: What are the relationships, if any, among $(a+b)^5$, the five-tall towers problem, the five-topping pizza problem, and the fifth row of Pascal's Triangle? This question is significant in terms of the students' later work, as it represents the first time the students were asked to think about a four-way link, among the binomial expansion, the two combinatorics problems, and Pascal's Triangle. The students were able to make the connection, evoking and expanding Ankur's explanation from the previous month of how the four-tall towers problem could be found in row 4 of Pascal's Triangle. (This also anticipated their night session explanation of entries in Pascal's Triangle in terms of pizzas.) In the following excerpt, the students linked the binomial expansion to the towers problem.

RESEARCHER:	What are the *a*'s and the *b*'s here?
ROMINA:	Colors. …
ANKUR:	*a* and *b* is red and blue. …
RESEARCHER:	What do you mean by red and blue?
ANKUR:	*a* is red and *b* is blue. That's [red-blue tower] *a b*. So *b a* would be a blue red.
RESEARCHER:	So how, if you have them in front of you, how would they look different?
ANKUR:	Red and blue, red's on top, and blue's on the bottom. Blue's on top and red's on the bottom.

In the following episode, Ankur explains how to find the answers to the five-tall towers problem in row 5 of Pascal's Triangle, and then Jeff and Romina locate the pizza answers in row 6. (Row 6 of Pascal's Triangle contains the numbers 1 6 15 20 15 6 1.)

ANKUR:	This [1] is no red.
JEFF:	Yeah.
ANKUR:	So there's one with no red. There's six with one red. … There's fifteen with two reds. Twenty with three reds. Six with five reds.
JEFF:	And one with no-
ANKUR:	And one with no-
ROMINA:	No.
ANKUR:	No. Six reds.
JEFF:	One with six reds. … All right. Now. What does that have to do with pizza?
ANKUR:	Just relate the tower problem to the pizza problem.
JEFF:	Well, we're saying that this [1] is a pizza with just plain.
ROMINA:	Yeah. That'll be the plain pizza.
JEFF:	Plain. This [6] is with all your six toppings.
ROMINA:	That's with one topping.
ANKUR:	You can't exactly relate these numbers to the pizza problem.
JEFF:	Well, we'll try really quick.
ROMINA:	Yeah. You can. 'Cause this [1] is plain, just plain pizza.
ANKUR:	And what will the other 1 represent?
ROMINA:	With everything on it.
ANKUR:	Okay.
JEFF:	So this is plain.
ANKUR:	Okay. Six with-
JEFF:	With one of each. Fifteen is with-
ROMINA:	Two toppings.

JEFF:	Just two toppings out of your six. Twenty is with three toppings. Fifteen is with the four toppings. Six is with the five toppings.
ROMINA:	Five toppings.
JEFF:	And the other one is-
ROMINA:	And the one is with all of them.
JEFF:	Like the supreme.
ROMINA:	Is that good?
ANKUR:	Cool. We're on fire today.

Thus Ankur, Jeff, and Romina used the two combinatorics problems they knew in order to explain the numbers in Pascal's Triangle. This was the first time they were observed connecting the pizza problem to Pascal's Triangle. Although Ankur had initially been reluctant to attempt a definition of the relationship between Pascal's Triangle and the pizza problem ("You can't exactly relate these numbers to the pizza problem"), he still participated in the discussion and at the end expressed satisfaction with their work. ("We're on fire today.") We noted earlier that the students received no special concrete rewards for participation in the study. Ankur's remarks illustrate our belief that the intellectual enjoyment involved with solving difficult problems was a factor in the students' continuing involvement in the study.

After explaining the link between specific numbers in Pascal's Triangle and the pizza and towers problems, the students described an instance of the addition rule in terms of towers problems. They explained the instance of Pascal's Identity shown in Figure 10-7 below. They described the two 10's in row 5 and of the 20 in row 6 as counting classes of five-tall towers, and they described the process by which the 20 (counting a different class of six-tall towers) could be generated from the two 10's. The transcript below gives portions of their discussion.

RESEARCHER:	What are those tens counting? And what does the twenty count?
JEFF:	The tens show-
ANKUR:	The tens show two of one color.
ROMINA:	And three of another.
ANKUR:	One color and two of another color. … That's why it's ten and ten. But then, at the top of each one, you can put either-
JEFF:	You could either put a red or like blue.
RESEARCHER:	The first ten in that row of five high has two reds and three blues? We're counting reds?
ANKUR:	Yes.
RESEARCHER:	And the second ten has-
ANKUR:	Three reds and-
RESEARCHER:	-three reds and two blues. Now coming down here, the twenty is supposed to count the ones that have three of each.
ANKUR:	Three red. Three reds and three blues.
JEFF:	Right.
RESEARCHER:	So how do the two tens add to give the twenty?
ANKUR:	Because in these ten, where there's three reds and two blues, you want to make it three reds and three blues. So you put a blue on top of each one.

This is the first instance where a connection was made between Pascal's Identity and a specific concrete combinatorics problem.

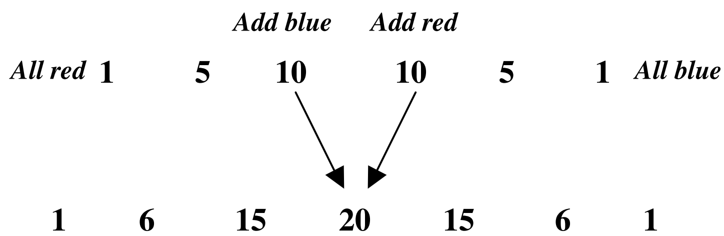

Figure 10-7. One instance of Pascal's Identity which students linked to towers problems

10.6 Session 5: Towers, Pizzas, and Pascal's Triangle

Ankur, Jeff, Mike, and Romina attended this session, four weeks after session 3. Much of the session was devoted to attempts by Ankur, Jeff, and Romina to explain to Mike specific instances of Pascal's Identity in terms of towers and the binomial expansion. This session also provided another example of the importance of revisiting problems and re-explaining solutions. In the February session, described above, the students had successfully explained the addition rule shown in Figure 10-7 in terms of towers: to create the 20 six-tall towers that have three red and three blue cubes, add a blue cube to the 10 five-tall towers with two blue cubes and add a red cube to the 10 five-tall towers with two red cubes. But this time, when asked to explain the similar case shown in Figure 10-8, they did not recall the previous (correct) explanation. Instead, they tried to find a solution that would involve disassembling and re-assembling existing towers, an approach that surprised and confounded the researcher.

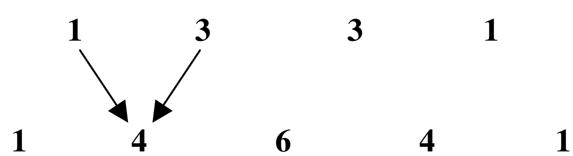

Figure 10-8. Another instance of Pascal's Identity discussed in terms of towers problems

They correctly mapped the 1 and 3 into towers (1 = white-white-white and 3 = blue-white-white, white-blue-white, and white-white-blue, as shown in Figure 10-9).

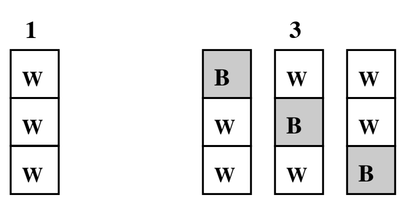

Figure 10-9. Students map specific towers to 1 and 3 in row 3 of Pascal's Triangle

But they tried to create the 4 by breaking apart the tower representing 1 and distributing its cubes among the other three towers. After the researcher questioned this method, Mike gave a different explanation for how to create the 4.

JEFF:	We've got this [the white-white-white tower, representing 1]. And we're saying how this goes together. [Jeff has assembled the three towers each with one blue cube to represent the 3. Refer to Figure 17.] We're saying- [Jeff starts to dismantle the white-white-white cube.]
RESEARCHER:	No. No. Don't take that apart. Because-
JEFF:	Well, that's why I made this. So I could.
ANKUR:	We made another one so we can take that one apart. … And show you.
RESEARCHER:	You mean, you mean, you mean you get the four by taking something apart?
ANKUR:	You're not taking it apart.
ROMINA:	You're not taking it apart; you're just seeing how they go together. …
MIKE:	You don't really have to take it apart to show this, 'cause look. Each one, the reason why they combine, each one of these four blocks [towers] is going to have something added to them to equal the same thing.
ANKUR:	Yeah.
MIKE:	These blocks [towers] are going to have, they're going to have a white block added to them. [Mike indicates the three three-tall towers with one blue cube.]
ANKUR:	They're going to have a *b* added to them.
MIKE:	And this one's [the white-white-white] going to have a, a blue added to it.
ANKUR:	An *a* added to it.
MIKE:	And they're going to equal the same thing. That's why you're going to have the four. [Refer to Figure 10-10 for a diagram of Mike and Ankur's suggestions.]

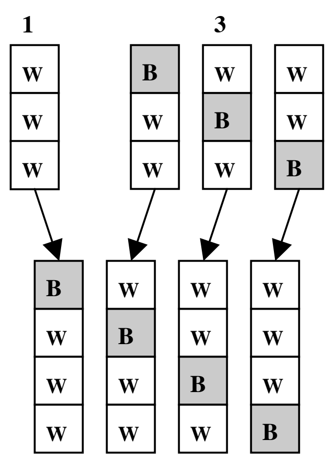

Figure 10-10. Mike and Ankur illustrate 1+3=4 by adding cubes to towers

The other students accepted Mike's explanation and apparently this time they comprehended the process, as evidenced by their later work in the night session (discussed in Chapter 12) and subsequent interviews. In addition, Ankur reiterated a link noted in the February session, observing that the *a*'s and *b*'s in the binomial expansion could be connected to the blue and white cubes, respectively, in the towers. The same connection was made a year later during the night session.

Next, the researcher asked the students to relate the towers problem to binary notation and the pizza problem; she said, "If you had to make up a pizza problem to model this row [row 2 of Pascal's Triangle], what's the pizza problem?" Ankur reiterated the position that he had taken in two previous sessions, that a peppers and pepperoni pizza is the same as a pepperoni and peppers pizza; it appeared that he had not yet firmly established that there was a connection between the two choices of cube colors for each cube in a tower and the two choices for each topping – on or off the pizza. Although the group noted that the n^{th} row of Pascal's Triangle could be linked to the *n*-topping pizza problem, they did not propose an explanation about how to use the numbers in the n^{th} row to enumerate pizzas. When the researcher asked for clarification and Ankur insisted that there was no relationship between colors and pizza toppings, Mike interrupted with his own explanation. It is interesting to note the similarity between this episode and the earlier one where the students discussed how to connect pizza problems to Pascal's Triangle. As in that episode, Ankur initially denied a connection. And as in that episode, when the connection was established, Ankur, with Jeff and Romina, quickly caught on and proceeded as enthusiastic participants in the exploration and explanation process. In this episode, also, we see a student express satisfaction with the group's intellectual achievement. (Romina said, "Oh, wow!") Figure 10-11 illustrates the four two-tall towers the group made as part of this process.

RESEARCHER: Now wait. Now I'm lost again. What, what, what was this? … [The Researcher indicates the single white and blue cubes representing row 2 of Pascal's Triangle.]
ANKUR: The colors don't, don't look at the colors.
MIKE: No. No. No.
ANKUR: Just look at this [Pascal's Triangle]. … But the colors don't specifically represent anything.
ROMINA: Yeah.
MIKE: Yes. It does.
ANKUR: No, it don't.
MIKE: Topping. [Mike points to the blue cube.] Or no topping. [Mike points to the white cube]. Just say like that. And if you look at it like this, you know.
ANKUR: So all of the whites are no topping?
MIKE: Yeah. [Mike takes the white-white-white tower.] Then this is a plain pizza with a choice. If you had a choice of three toppings.
JEFF: All right.
ANKUR: Okay.
ROMINA: Okay.
MIKE: This [the blue-white-blue tower] would be a pizza-
ROMINA: Oh. With the one. Ooh.
MIKE: -with two different toppings, without the other, third topping.
ROMINA: That's what I was asking.
ANKUR: Okay.
JEFF: … Well, yeah. Well, if you're just saying that this [the white-white-white tower] is the pizza with three no toppings, it's plain.
ROMINA: It's just a plain pizza.
ANKUR: All right. All right. So that's [blue-blue tower] two toppings.
ROMINA: Yeah.
JEFF: Yeah. All right. So.
MIKE: That's [white-white tower] … a choice of two, but you want it plain.
ANKUR: You have a choice of two toppings.
JEFF: Yeah, so this is, this [blue-blue tower] is choice of two using two. This [blue-white tower] is choice of two using one.
ANKUR: Two using one.
JEFF: This [white-blue tower] is choice of two using the other one.
ANKUR: That's using the other one. And that's [white-white tower] using nothing.
ROMINA: Yeah.
RESEARCHER: And that's all the possibilities?
ANKUR: Yes.
ROMINA: Yeah.

RESEARCHER: You like that?
ROMINA: Oh, wow!

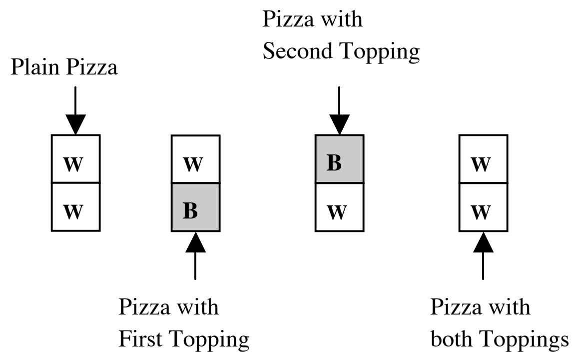

Figure 10-11. The students' link between two-tall towers and two-topping pizzas

During this episode, the other three members of the group immediately accepted and built upon Mike's brief remarks. All he had to say was "topping" and "no topping," and all three of the others began immediately to form connections between specific individual towers and specific pizzas. This represented the fifth discussion of the pizza problem in four months, and at least three members of the group apparently began this discussion without a clear idea of the essential feature of the problem (topping vs. no topping), as opposed to a surface feature (the fact that the toppings could be selected in any order). But it appears that this discussion helped them finally to make sense of the isomorphic relationship, because the pizza problem was the one that the group selected during the night session a year later, to explain Pascal's Identity.

Figure 10-12. Michael shows that blue means "topping" and white means "no topping"

10.7 Discussion

During these five tenth-grade problem solving sessions, the students worked independently, sometimes spontaneously splitting themselves into subgroups, sometimes working individually, but always sharing their ideas with the other members of the group. By sharing, the students were able to incorporate others' ideas into their own understanding of the justifications. An example is Mike's introduction of the binary notation code. Mike watched Jeff and Romina work for a while, listened as the other four students exchanged ideas, and then began focusing on his own paper. He later presented his conception of how the binary system mapped onto the solution of the pizza problem. Mike recalled an episode from an eighth grade class and applied this previously constructed knowledge to this totally new situation. His introduction of a coding system, the zeroes and ones of the binary system, to the justification being built by the group of five students was an important contribution. It became the students' notation of choice for future problems.

Over the course of these sessions, we observed the students investigating problems that had been exploring in earlier years, retrieving earlier ideas and images as they built solutions and justifications. These ideas and images sometimes reappeared just as they were formed in the prior occurrences. In the third session of the sophomore year, we see Mike and Ankur's swift production of a justification for the number of five-tall towers with exactly two red cubes. They reproduced a justification by cases that had originally been built in their fifth-grade classroom, but using Mike's binary notation. In addition, they offered a second justification, utilizing a strategy that depended on controlling for variable that was first introduced by Ankur while solving a pizza problem in grade five. His ownership of this strategy allowed him to adapt it for use in the isomorphic block tower problem.

For the same problem, Jeff retrieved a strategy used during that same fifth-grade session. Mike and Ankur had enthusiastically participated in the whole classroom discussion, which culminated in a proof by cases. Although Jeff had been in the room, he was not focused on the classroom discussion; he was looking at patterns in the towers that he had built. Jeff's partners, Romina and Brian, also had more difficulty in providing a justification during the session in their sophomore year. In grade five, while Mike and Ankur were active participants in the classroom discussion and Jeff was quietly pursuing his own line of thinking, Brian and Romina were in another classroom. Although they worked on the five-tall towers problem, their class did not offer a convincing justification for the answer to that problem. The difficulties experienced by these students as they worked on the block tower problems as tenth graders might be explained by the absence of some earlier experiences. They constructed the images and representations to the block tower problem for the first time in this tenth grade experience.

In the attempt to think about the potential connection between the pizza and block tower problems, the students came to discuss many powerful mathematical concepts. While they were able at an early point to determine that the answer to the n-topping pizza problem is 2^n, they came to this number by recognizing the pattern of {2, 4, 8, 16, 32, …}. They determined that n represented the number of toppings, but did not provide a satisfactory explanation for the base 2 until they began the discussing the possible relationship between pizzas and block towers; then that they came to see that the base 2 represented the presence or absence of a topping. This realization came, not from working on the pizza problem, but instead as a result of their search for an answer to the three-color four-tall towers problem. Thus we see that the opportunity to work on open-ended problems and follow paths determined by the interest of the moment can lead to greater understanding of other problems. In this case, the opportunity to investigate an isomorphic problem provided the students with the tools necessary to complete the formulation of the imperfectly developed earlier idea.

In summary, these students investigated isomorphic problems in combinatorics and used them to explore how Pascal's Triangle grows and to make sense of Pascal's Identity. Between December 1997 and March 1998, they first found general solutions to the pizza and towers problems, using letter and number codes and binary notation to enumerate the pizzas and towers. Then they organized their lists of solutions, organizing the pizza problem solutions according to number of toppings and the towers problems solutions according to the number of cubes of one color. These lists not only provided a way to show that all cases were present, but they also provided the means to associate those cases with the numbers in Pascal's Triangle. In discussions with the researcher and other researchers, these students described the isomorphic relationship between the pizza and towers problems. Their extensive repertoire of representations proved essential; in this process, they made use of words, written inscriptions, and concrete materials (as when Mike held up a blue cube and a white cube and said "topping" and "no topping"). The opportunity to revisit problems also proved crucial, as students often needed to have two, three, or more discussions on the same topic before critical ideas were firmly established.

In the next chapter, we observe another cohort of students who also make sense of the relationships among towers, pizzas, and Pascal's Triangle. They brought their own experience, their own representations, and their own ideas to the problem, but they too used personal representations, communicated findings, and made generalizations that showed their increased understanding of these problems in combinatorics.